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A comparison of different tests

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Publication date:
2003

Document version
Publisher's PDF, also known as Version of record

Citation for published version (APA):
Madsen, E. (2003). *Unit root inference in panel data models where the time-series dimension is fixed: A comparison of different tests*. Department of Economics, University of Copenhagen.

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2003-13

The activities of CAM are financed by a grant from
The Danish National Research Foundation

Unit root inference in panel data models where the time-series dimension is fixed: A comparison of different tests

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August 2003

Abstract

The objective of the paper is to investigate and compare the performance of some of the unit root tests in micro-panels which have been suggested in the literature. The framework is an autoregressive panel data model allowing for heterogeneity in the intercept but not in the autoregressive parameter. The tests being considered can be used to distinguish between the null hypothesis of each time-series process being a random walk and the alternative hypothesis of each time-series process being stationary with individual-specific levels but the same autoregressive parameter. In addition, the tests are all based on usual t-statistics corresponding to least squares estimators of the autoregressive parameter resulting from different transformations of the model. The performance of the tests is investigated by deriving the local power of the tests when the autoregressive parameter is local-to-unity. The results show that the assumption concerning the initial values is important in this matter. The outcome of a simulation experiment demonstrates that the local power of the tests provides a good approximation to their actual power in finite samples.

Keywords: Dynamic panel data model; Unit roots; Local alternatives; Initial values

JEL classification: C12; C23

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1 Introduction

In this paper we investigate unit root inference in panel data models where the cross-section dimension is much larger than the time-series dimension. So we consider traditional micro-panels. At present there is a large econometric literature dealing with unit root testing in panel data models which has developed during the last ten years. Contrary to the previous literature on dynamic panel data models, a large part of this new literature considers macro-panels where the cross-section and time-series dimensions are similar in magnitude. Banerjee (1999) and Baltagi & Kao (2000) review many of the contributions to the literature on unit root testing in panel data models. Reviews of the literature on dynamic micro-panels are provided in Hsiao (1986), Baltagi (1995) and recently Arellano (2003) of which only the latter discusses the issue of unit roots.

The analysis in this paper is done within the framework of a first-order autoregressive panel data model allowing for individual-specific levels. This means that we are testing the null hypothesis of each time-series process being a random walk without drift against the alternative hypothesis of each time-series process being stationary with individual-specific levels but the same autoregressive parameter for all cross-section units. This means that the model does not allow for individual-specific linear time trends. In the autoregressive panel data model there are two sources of persistency. One is the autoregressive mechanism which is the same for all cross-section units and the other is the unobserved individual-specific term. The unit root hypothesis can then be considered as an extreme case where all persistency is caused by the autoregressive mechanism. The hypothesis is of interest since many economic variables at the individual level, such as income of individuals and firm level variables, are found to be persistent over time. For a discussion of this issue see Section 5 in Arellano (2003).

The main contribution of the paper is to provide analytical results about the performance of some of the unit root tests which have been suggested in the literature. More specifically, to provide analytical results about the asymptotic power of these tests when the value of the autoregressive parameter is close to unity. This is done by deriving the limiting distributions of the corresponding test statistics under local alternatives when the autoregressive parameter is local-to-unity. The results are used to compare the performance of different tests in terms of their local power. In addition they reveal how the local power of the tests is affected by the nuisance parameters of the data generating process (DGP). Until now the power properties of unit root tests in micro-panels have only been investigated and compared in simulation studies, see for example Bond, Nauges & Windmeijer (2002) and Hall & Mairesse (2002). However, the outcome of these might depend on the particular choice of nuisance parameters in the simulation setup in a non-transparent way. Therefore, it seems to be a useful contribution within this research area. The paper by Breitung (2000) is related to this paper as it investigates the local power of some of the unit root tests in macro-panels.

We consider three different unit root tests. The corresponding test statistics are all t-statistics based on least squares (LS) estimators of the autoregressive parameter which result from different transformations of the autoregressive panel data model. The transformations are the following: (i) the original model, (ii) the model with variables expressed in terms of deviations from the initial values, and (iii) the model with variables expressed in terms of deviations from individual-specific time-series means. The unit root test corresponding to (i) is suggested by Bond, Nauges & Windmeijer (2002) and is based on the OLS estimator of the autoregressive parameter. The reason for choosing this statistic is that the OLS estimator in spite of being inconsistent under the alternative hypothesis is consistent under the null hypothesis of a unit root. Under the alternative hypothesis the inconsistency is caused by the individual-specific terms. The unit root tests corresponding to (ii) and (iii) are suggested by Breitung & Meyer (1994) and Harris & Tzavalis (1999), respectively. Contrary to the OLS estimator, the LS estimators corresponding to (ii) and (iii) are invariant with respect to the individual-specific levels. Clearly, this leads to tests which are invariant with respect to individual-specific levels even in finite samples. It means that under the null hypothesis they are invariant with respect to the initial values and under the mean stationary alternative they are invariant with respect to the unobserved individual-specific terms. In particular, the LS estimators corresponding to (ii) and (iii) do not have an asymptotic bias caused by the individual-specific terms. Instead they suffer from different types of biases. Also by performing these transformations of the variables there might be a loss of precision in the corresponding LS estimators.

The LS estimator of the autoregressive parameter corresponding to (ii) is consistent under the null hypothesis and inconsistent under the covariance stationary alternative. It is well-known that the within-group estimator of the autoregressive parameter corresponding to (iii) is inconsistent under the covariance stationary alternative and suffers from the so-called Nickell-bias, see Nickell (1981). Under the null hypothesis Harris & Tzavalis (1999) show that this is also the case but that the asymptotic bias does not depend on any nuisance parameters but instead is a function of the time-series dimension of the panel. Therefore they suggest to use the bias adjusted within-group estimator. In addition, they show that the limiting variance of the bias adjusted within-group estimator only depends on the time-series dimension of the panel. So instead of the t-statistic they consider a normalized coefficient statistic. However, the expression for the limiting variance is only valid when strong assumptions are imposed on the errors. As the usual t-statistic does not rely on such strong assumptions, we suggest to use this statistic.

From the description above, it is not straightforward to determine which test is best in terms of having the highest power. The results in this paper show that the asymptotic power of the tests under local alternatives differs depending on the assumption being made about the initial values, i.e. whether these are such that the time-series processes become mean stationary or covariance stationary. Altogether, the results show that the local power of the Breitung-Meyer test is always higher than the local power of the Harris-Tzavalis test. Furthermore, in some situations the local power of the OLS test is higher than the local power of the Breitung-Meyer test. This is most likely to be the case when the time-series processes are covariance stationary.

The paper is organized as follows. In Section 2, the basic model is specified. In Section 3, we investigate and compare the three different unit root tests described above. This is done by deriving the limiting distributions of the corresponding test statistics under local alternatives. In Section 4, the analytical results are illustrated in a simulation study. In Section 5, we provide some concluding remarks.

2 The model and assumptions

We consider the first-order autoregressive panel data model with individual-specific intercepts defined by

$$y_{it} = \rho y_{it-1} + (1-\rho)\,\alpha_i + \varepsilon_{it} \quad \text{for } i = 1, ..., N \text{ and } t = 1, ..., T \tag{1}$$

where $-1 < \rho \leq 1$ and for every $i = 1, ..., N$ the sequence $\{\varepsilon_{it}\}_{t=1}^{\infty}$ is white noise. For notational convenience we assume that the initial values y_{i0} are observed such that the actual number of observations over time equals $T+1$. The model provides a framework for testing the null hypothesis of each time-series process being a random walk against the alternative hypothesis of each time-series process being stationary with an individual-specific level. To specify the model further the assumptions below are imposed.

Assumption 1 *ε_{it} is independent across i, t with $E(\varepsilon_{it}) = 0$, $E\left(\varepsilon_{it}^2\right) = \sigma_{i\varepsilon}^2$ and $E\left(\varepsilon_{it}^4\right) = E\left(\varepsilon_{is}^4\right)$ for all $t, s = 1, ..., T$. In addition ε_{it} is independent of α_i and y_{i0}.*

Assumption 2 *α_i is iid across i with $E(\alpha_i) = 0$, $E\left(\alpha_i^2\right) = \sigma_\alpha^2$ and $E\left(\alpha_i^4\right) < \infty$.*

Assumption 3 *For $-1 < \rho \leq 1$ the initial values satisfy $y_{i0} = \mathbf{1}_{\{|\rho|<1\}}\alpha_i + \sqrt{\tau(\rho)}\varepsilon_{i0}$ where ε_{i0} is independent of α_i and independent across i with $E(\varepsilon_{i0}) = 0$ and $E\left(\varepsilon_{i0}^2\right) = \sigma_{i\varepsilon}^2$. The scaling function $\tau(\rho)$ can be on the following forms: (i) $\tau(\rho) = \tau$ for $0 \leq \tau < \infty$ when $-1 < \rho \leq 1$, (ii) $\tau(\rho) = 1/\left(1-\rho^2\right)$ when $-1 < \rho < 1$.*

Assumption 4 *The following hold:*

(i) $E|\varepsilon_{it}|^{4+\delta} < K < \infty$ *for some* $\delta > 0$ *and all* $i = 1, ..., N,\ t = 0, 1, ..., T$

(ii) $\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^2 \to \sigma_{2\varepsilon} > 0$ *as* $N \to \infty$

(iii) $\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^4 \to \sigma_{4\varepsilon}$ *as* $N \to \infty$

(iv) $\frac{1}{N}\sum_{i=1}^{N}E\left(\varepsilon_{it}^4\right) \to m_4$ *as* $N \to \infty$

Assumption 1 states that the errors ε_{it} are independent over cross-section units and time and allowed to be heteroskedastic over cross-section units but not over time. Further, they are independent of the individual-specific term α_i and the initial value y_{i0}. The assumption about independency over time is stronger than the usual assumption about ε_{it} being serially uncorrelated. It is a simplifying assumption made in order to derive the asymptotic properties of the test statistics in Section 3. Assumption 2 concerning α_i is standard in dynamic panel data models. Assumption 3 specifies the initial values. When $\rho = 1$ the assumption states that the initial values have finite variance. When $|\rho| < 1$ the assumption implies that initial values are such that the time-series processes for y_{it} become mean stationary and we consider two different assumptions about the dispersion of the initial deviation from the stationary level. In (i) where $\tau(\rho) = \tau$ the dispersion does not depend on the autoregressive parameter whereas in (ii) where $\tau(\rho) = 1/\left(1-\rho^2\right)$ it does. In the latter case the time-series processes become covariance stationary. The distinction appears to be very important for the results in Section 3. Note that ε_{it} is independent of ε_{i0} by Assumption 1. Finally, Assumption 4 is a technical assumption which enables us to derive the asymptotic properties of the statistics of interest by applying standard asymptotic theory. The assumption states that the innovations ε_{it} have uniformly bounded moments of order slightly greater than four and that the cross-section average of their variances, squared variances and fourth order moments have well-defined limits as the cross-section dimension N tends to infinity. Note that when the errors ε_{it} are homoskedastic across units then $\sigma_{4\varepsilon} = \sigma_{2\varepsilon}^2$. Assumption 4 (iv) is only required in relation to the test statistic suggested by Harris & Tzavalis (1999), as this is the only statistic of the ones considered in this paper which depends on fourth order moments.

3 The test statistics and their asymptotic properties

We consider the testing problem where the null hypothesis and the alternative hypothesis are given by

$$H_0 : \rho = 1 \qquad H_A : |\rho| < 1 \tag{2}$$

In the following we consider local alternatives where ρ is modelled as being local-to-unity. More specifically, we consider local-to-unity sequences for ρ defined by

$$\rho = 1 - \frac{c}{N^k} \quad \text{for } k, c > 0 \tag{3}$$

This means that as the sample size N increases, the value of the parameter ρ is in a N^{-k} neighborhood of unity. So instead of deriving asymptotic representations based on ρ being constant as N increases we derive asymptotic representations based on $c = (1-\rho) N^k$ being constant as N increases. The idea is that these representations will provide good approximations to the actual distributions of the relevant statistics. With one exception the LS estimators of ρ considered in this paper converge weakly to normal distributions at the rate $\sqrt{N}$ and therefore we consider local-to-unity sequences for ρ with $k = \frac{1}{2}$. In one situation, the LS estimator must be normalized differently in order to converge weakly to a non-degenerate distribution under the local alternative and the local-to-unity sequence is defined accordingly. Note that $c = 0$ corresponds to the null hypothesis of ρ being unity.

It appears that the limiting distributions of the different statistics under the local alternative defined by (3) depend on the assumption being made about the dispersion of the initial deviation from the stationary level. According to Assumption 3, we consider the following two situations

$$(i) \text{ and } -1 < \rho \leq 1 : \qquad \tau(\rho) = \tau < \infty \tag{4}$$

$$(ii) \text{ and } -1 < \rho < 1 : \qquad \tau(\rho) = \frac{1}{1-\rho^2} \tag{5}$$

Under the local-to-unity sequence for ρ given by $\rho = 1 - c/N^k$ this corresponds to

$$(i) \text{ and } c \geq 0: \qquad \tau(\rho) = \tau = O(1) \tag{6}$$

$$(ii) \text{ and } c > 0: \qquad \tau(\rho) = \frac{N^k}{2c} + o(N^{-k}) \tag{7}$$

where (7) holds according to Lemma 1 in Appendix A.1. When the initial values are such that the time-series processes become mean stationary but not covariance stationary as given by (6), the variability of the variable y_{it} is bounded as N tends to infinity. On the other hand, when the initial values are such that the time-series processes become covariance stationary as given by (7), the variance of the initial deviation from the stationary level is of order N^k and hence this term dominates the behavior of the variable y_{it} as N tends to infinity. Under the null hypothesis, the variance of the initial value and by that the variance of y_{it} is bounded as N tends to infinity. Hence, the asymptotic behavior of y_{it} is similar under the mean stationary local alternative and under the null hypothesis of ρ being unity but different under the covariance stationary local alternative.

3.1 OLS

The equation in (1) can be rewritten as the following regression model

$$\begin{aligned} y_{it} &= \rho y_{it-1} + u_{it} \\ u_{it} &= (1-\rho)\,\alpha_i + \varepsilon_{it} \end{aligned} \qquad \text{for } i = 1, ..., N \text{ and } t = 1, ..., T \tag{8}$$

The OLS estimator of the autoregressive parameter ρ is defined by

$$\hat{\rho}_{OLS} = \left(\sum_{i=1}^{N} y'_{i,-1} y_{i,-1}\right)^{-1} \left(\sum_{i=1}^{N} y'_{i,-1} y_i\right) \tag{9}$$

where $y_i = (y_{i1}, ..., y_{iT})'$ and $y_{i,-1} = (y_{i0}, ..., y_{iT-1})'$. The estimator is consistent when $\rho = 1$ whereas inconsistent when $|\rho| < 1$. In the latter case, the inconsistency is attributable to the term α_i which appears in both the regressor y_{it-1} and the regression error u_{it}. As α_i appears with the factor $(1-\rho)$ in u_{it} the covariance between the regressor and the regression error is positive and decreases towards zero as ρ approaches unity. Now the regressor y_{it-1} can be expressed as the sum of the two independent terms α_i and $(y_{it-1} - \alpha_i)$ which are the stationary level and the deviation from the stationary level, respectively. If the variability of the two terms are of similar order as ρ approaches unity, the asymptotic bias of $\hat{\rho}_{OLS}$ is positive and decreases towards zero as ρ approaches unity. This describes the situation where the initial values are such that the time-series processes become mean stationary but not covariance stationary. On the other hand, if the behavior of y_{it-1} is dominated by the term $(y_{it-1} - \alpha_i)$ as ρ approaches unity, the asymptotic bias of $\hat{\rho}_{OLS}$ will be zero when ρ approaches unity. This describes the situation where the initial values are such that the time-series processes become covariance stationary.

The discussion above is formalized by the results given in Proposition 1 below. The proposition provides the limiting distribution of the OLS estimator $\hat{\rho}_{OLS}$ under both the null hypothesis when ρ is unity and local alternatives when ρ is local-to-unity. We consider different local alternatives depending on the assumption being made about the initial values as given by (i) and (ii) in Assumption 3.

Proposition 1 *Under Assumption 1, 2, 3(i), 4 and the local-to-unity sequence for ρ given by $\rho = 1 - c/\sqrt{N}$ for $c \geq 0$, the limiting distribution of the OLS estimator $\hat{\rho}_{OLS}$ is given by*

$$\sqrt{N}\left(\hat{\rho}_{OLS} - \rho\right) \xrightarrow{w} N\left(c\frac{\sigma_\alpha^2}{\sigma_\alpha^2 + \left(\tau + \frac{T-1}{2}\right)\sigma_{2\varepsilon}}, \frac{1}{T}\frac{\mathbf{1}_{\{c>0\}}\sigma_\alpha^2\sigma_{2\varepsilon} + \left(\tau + \frac{T-1}{2}\right)\sigma_{4\varepsilon}}{\left(\mathbf{1}_{\{c>0\}}\sigma_\alpha^2 + \left(\tau + \frac{T-1}{2}\right)\sigma_{2\varepsilon}\right)^2}\right) \quad \text{as } N \to \infty \tag{10}$$

Under Assumption 1, 2, 3(ii), 4 and the local-to-unity sequence for ρ given by $\rho = 1 - \tilde{c}/N$ for $\tilde{c} > 0$, the limiting distribution of the OLS estimator $\hat{\rho}_{OLS}$ is given by

$$N\left(\hat{\rho}_{OLS} - \rho\right) \xrightarrow{w} N\left(0, \tilde{c}\frac{\sigma_{4\varepsilon}}{\sigma_{2\varepsilon}^2}\frac{2}{T}\right) \quad \text{as } N \to \infty \tag{11}$$

The proof of Proposition 1 is given in Appendix A.2. The proposition shows that in the unit root case when $c = 0$, the estimator $\hat{\rho}_{OLS}$ is $\sqrt{N}$-consistent and its limiting variance is decreasing in τ, T and $\sigma_{2\varepsilon}^2/\sigma_{4\varepsilon}$. As indicated above, it turns out that the asymptotic behavior under local alternatives is very different depending on the assumption concerning the initial values. When the initial values are such that the time-series processes become mean stationary, the estimator $\hat{\rho}_{OLS}$ has an asymptotic bias of order $1/\sqrt{N}$ under the local alternative. The bias is always positive and increasing in $\sigma_\alpha^2/\sigma_{2\varepsilon}$ and c (i.e. for a fixed N it is decreasing in ρ at the rate $1/\sqrt{N}$) and decreasing in τ and T. The limiting variance of $\hat{\rho}_{OLS}$ is decreasing in τ, T and $\sigma_{2\varepsilon}^2/\sigma_{4\varepsilon}$ and increasing in $\sigma_\alpha^2/\sigma_{2\varepsilon}$. On the other hand, when the initial values are such that the time-series processes become covariance stationary, the estimator $\hat{\rho}_{OLS}$ is N-consistent under the local alternative. This means that $\hat{\rho}_{OLS}$ estimates the parameter ρ very precisely when its true value is close to unity. Further, the limiting variance of $\hat{\rho}_{OLS}$ is increasing in $\tilde{c}$ (i.e. for a fixed N it is decreasing in ρ at the rate $1/N$) and decreasing in T and $\sigma_{2\varepsilon}^2/\sigma_{4\varepsilon}$. This rather surprising result is explained as follows. When ρ is local-to-unity, the behavior of y_{it} for $t = 0, ..., T$ is dominated by the initial deviation from the stationary level $(y_{i0} - \alpha_i)$. More specifically, the variance of $(y_{i0} - \alpha_i)$ is of order N under the local-to-unity sequence for ρ given by $1 - \tilde{c}/N$ for $\tilde{c} > 0$, see the result in (7), whereas the variance of the remaining terms in y_{it} is bounded as N tends to infinity. This implies that the numerator in (9) must be normalized by N in order to converge in distribution and the denominator in (9) must be normalized by N^2 in order to converge in probability. The consistency is a result of the term $(y_{i0} - \alpha_i)$, which dominates the behavior of the regressor, being independent of the term α_i. This indicates that the asymptotic representation in (11) is only appropriate when the variances of α_i and ε_{it} are much smaller than the variance of $(y_{i0} - \alpha_i)$. Once the variances are of similar magnitude, the asymptotic representation in (10) is expected to provide a better approximation to the actual distribution of $\hat{\rho}_{OLS}$.

The unit root test based on the usual t-statistic is obtained by normalizing $(\hat{\rho}_{OLS} - 1)$ appropriately. For this purpose we need a consistent estimator of the limiting variance of $\hat{\rho}_{OLS}$ and we use White's heteroskedastic consistent estimator, see White (1980). Under the covariance stationary local alternative this estimator must be normalized differently in order to be consistent. Letting $k = \frac{1}{2}$ and $k = 1$ refer to the situations where $\hat{\rho}_{OLS}$ converges in distribution at the rate $\sqrt{N}$ and N respectively, White's heteroskedastic consistent estimator of the limiting variance of $\hat{\rho}_{OLS}$ is given by the following expression

$$\hat{V}_{OLS}(k) = \left(\frac{1}{N^{2k}} \sum_{i=1}^{N} y_{i,-1}' y_{i,-1} \right)^{-1} \frac{1}{N^{2k}} \sum_{i=1}^{N} y_{i,-1}' \hat{u}_i \hat{u}_i' y_{i,-1} \left(\frac{1}{N^{2k}} \sum_{i=1}^{N} y_{i,-1}' y_{i,-1} \right)^{-1} \tag{12}$$

where the vector of residuals is $\hat{u}_i = y_i - \hat{\rho}_{OLS} y_{i,-1}$. The t-statistic is then defined as

$$t_{OLS} = \hat{V}_{OLS}(k)^{-\frac{1}{2}} N^k (\hat{\rho}_{OLS} - 1) \tag{13}$$

The proposition below provides the limiting distribution of the t-statistic.

Proposition 2 *Under Assumption 1, 2, 3(i), 4 and the local-to-unity sequence for ρ given by $\rho = 1 - c/\sqrt{N}$ for $c \geq 0$, the limiting distribution of the OLS t-statistic t_{OLS} is given by*

$$t_{OLS} \xrightarrow{w} N\left(-c\left(\tau + \frac{T-1}{2}\right) \sqrt{\left(\frac{\sigma_\alpha^2}{\sigma_{2\varepsilon}} + \left(\tau + \frac{T-1}{2}\right) \frac{\sigma_{4\varepsilon}}{\sigma_{2\varepsilon}^2} \right)^{-1} T}, 1 \right) \quad as\ N \to \infty \tag{14}$$

Under Assumption 1, 2, 3(ii), 4 and the local-to-unity sequence for ρ given by $\rho = 1 - \tilde{c}/N$ for $\tilde{c} > 0$, the limiting distribution of the OLS t-statistic t_{OLS} is given by

$$t_{OLS} \xrightarrow{w} N\left(-\sqrt{\tilde{c} \frac{\sigma_{2\varepsilon}^2}{\sigma_{4\varepsilon}} \frac{T}{2}}, 1 \right) \quad as\ N \to \infty \tag{15}$$

The proof of Proposition 2 is given in Appendix A.2. The proposition shows that under the null hypothesis of a unit root, the t-statistic t_{OLS} is asymptotically standard normal. So unit root inference is carried out by employing critical values from the standard normal distribution. Furthermore, the proposition shows that under the mean stationary alternative, the local power is increasing in τ, T and $\sigma_{2\varepsilon}^2/\sigma_{4\varepsilon}$ (the location parameter is shifted to the left when these parameters increase) and decreasing in $\sigma_\alpha^2/\sigma_{2\varepsilon}$ (the location parameter is shifted to the right when $\sigma_\alpha^2/\sigma_{2\varepsilon}$ increases). Under the covariance stationary alternative, the local power only depends on T and $\sigma_{2\varepsilon}^2/\sigma_{4\varepsilon}$ and is increasing in both. As discussed above, the limiting distribution in (15) will only provide a good approximation to the actual distribution of the t-statistic when the behavior of y_{it} is dominated by the initial deviation from the stationary level.

Altogether, the advantage of using the OLS unit root test is that it is expected to have high power under the covariance stationary alternative even for values of ρ very close to unity. However, if the assumption about the time-series processes being covariance stationary is not valid, the power of the test for values of ρ close to unity is expected to be low when $\sigma_\alpha^2/\sigma_{2\varepsilon}$ is high. This will be most pronounced for small values of T.

3.2 Breitung-Meyer

Subtracting the initial value y_{i0} from both sides of the equation in (1) yields the following regression model

$$\begin{aligned} y_{it} - y_{i0} &= \rho (y_{it-1} - y_{i0}) + v_{it} \\ v_{it} &= (\rho - 1)(y_{i0} - \alpha_i) + \varepsilon_{it} \end{aligned} \qquad \text{for } i = 1, ..., N \text{ and } t = 1, ..., T \tag{16}$$

The LS estimator of ρ obtained from this regression equation is defined by

$$\hat{\rho}_0 = \left(\sum_{i=1}^{N} \tilde{y}_{i,-1}'\tilde{y}_{i,-1}\right)^{-1} \left(\sum_{i=1}^{N} \tilde{y}_{i,-1}'\tilde{y}_i\right) \tag{17}$$

where $\tilde{y}_i = y_i - y_{i0}\iota_T$, $\tilde{y}_{i,-1} = y_{i,-1} - y_{i0}\iota_T$ and ι_T is a $T \times 1$ vector of ones. Again the estimator is consistent when $\rho = 1$ whereas inconsistent when $|\rho| < 1$. In the latter case, its asymptotic bias equals $\frac{1}{2}(1-\rho)$ under the assumption about covariance stationarity, see Breitung & Meyer (1994). As an example, this means that the asymptotic bias equals 0.050, 0.025 and 0.005 when ρ equals 0.90, 0.95 and 0.99, respectively. The inconsistency is attributable to the term $(y_{i0} - \alpha_i)$ as it appears in both the regressor $(y_{it-1} - y_{i0})$ and the regression error v_{it}. The covariance between the regressor and the regression error decreases towards zero as ρ approaches unity when the variance of $(y_{i0} - \alpha_i)$ is kept constant. However, the decrease might be offset if the variance of $(y_{i0} - \alpha_i)$ increases as ρ approaches unity. This is exactly what happens when the initial values are such that the time-series processes become covariance stationary.

Proposition 3 below provides the limiting distribution of the Breitung-Meyer estimator $\hat{\rho}_0$ under both the null hypothesis when ρ is unity and the local alternative when ρ is local-to-unity. In this case, the local alternatives are the same irrespective of the assumption about the dispersion of the initial deviation from the stationary level.

Proposition 3 *Under Assumption 1, 2, 3, 4 and the local-to-unity sequence for ρ given by $\rho = 1 - c/\sqrt{N}$, the limiting distribution of the Breitung-Meyer estimator $\hat{\rho}_0$ is given by*

$$(i) \text{ and } c \geq 0 \quad : \quad \sqrt{N}\left(\hat{\rho}_0 - \rho\right) \xrightarrow{w} N\left(0, \frac{\sigma_{4\varepsilon}}{\sigma_{2\varepsilon}^2}\frac{2}{T(T-1)}\right) \quad \text{as } N \to \infty \tag{18}$$

$$(ii) \text{ and } c > 0 \quad : \quad \sqrt{N}\left(\hat{\rho}_0 - \rho\right) \xrightarrow{w} N\left(\frac{c}{2}, \frac{\sigma_{4\varepsilon}}{\sigma_{2\varepsilon}^2}\frac{2}{T(T-1)}\right) \quad \text{as } N \to \infty \tag{19}$$

The proof of Proposition 3 is given in Appendix A.3. The proposition shows that in the unit root case and under the mean stationary local alternative $\hat{\rho}_0$ is $\sqrt{N}$-consistent. Under the covariance stationary alternative $\hat{\rho}_0$ has a positive asymptotic bias of order $1/\sqrt{N}$. The limiting variance of $\hat{\rho}_0$ does not depend on the assumption being made about the initial values and it is a simple function of T and $\sigma_{2\varepsilon}^2/\sigma_{4\varepsilon}$ which is increasing in both. As indicated above, the results follow by using that when the variance of $(y_{i0} - \alpha_i)$ is of order less than $\sqrt{N}$, the asymptotic bias disappears under the local alternative. This is the case when the initial values are such that the time-series processes are mean stationary but not covariance stationary, see the result in (6). On the contrary, when the initial values are such that the time-series processes become covariance stationary this is not the case, as the variance of $(y_{i0} - \alpha_i)$ in this case is of order $\sqrt{N}$, see the result in (7).

As before White's heteroskedastic consistent estimator of the limiting variance of $\hat{\rho}_0$ is given by the following expression

$$\hat{V}_0 = \left(\frac{1}{N}\sum_{i=1}^{N} \tilde{y}_{i,-1}'\tilde{y}_{i,-1}\right)^{-1} \frac{1}{N}\sum_{i=1}^{N} \tilde{y}_{i,-1}'\hat{v}_i\hat{v}_i'\tilde{y}_{i,-1} \left(\frac{1}{N}\sum_{i=1}^{N} \tilde{y}_{i,-1}'\tilde{y}_{i,-1}\right)^{-1} \tag{20}$$

where the vector of residuals is $\hat{v}_i = \tilde{y}_i - \hat{\rho}_0 \tilde{y}_{i,-1}$. The t-statistic is then defined as

$$t_0 = \hat{V}_0^{-\frac{1}{2}} \sqrt{N} \left(\hat{\rho}_0 - 1\right) \tag{21}$$

When the errors ε_{it} are homoskedastic across units such that $\sigma_{4\varepsilon} = \sigma_{2\varepsilon}^2$, the limiting variance of $\hat{\rho}_0$ is a function of T only. Therefore, it is possible to use a normalized coefficient statistic when testing the unit root hypothesis. The statistic is defined in the following way

$$\bar{t}_0 = \sqrt{\frac{T\left(T-1\right)}{2}} \sqrt{N} \left(\hat{\rho}_0 - 1\right) \tag{22}$$

The proposition below provides the limiting distributions of the test statistics defined above.

Proposition 4 *Under Assumption 1, 2, 3, 4 and the local-to-unity sequence for ρ given by $\rho = 1 - c/\sqrt{N}$, the limiting distribution of the Breitung-Meyer t-statistic t_0 is given by*

$$(i) \text{ and } c \geq 0 \quad : \quad t_0 \xrightarrow{w} N\left(-c\sqrt{\frac{\sigma_{2\varepsilon}^2}{\sigma_{4\varepsilon}} \frac{T\left(T-1\right)}{2}}, 1\right) \quad \text{as } N \to \infty \tag{23}$$

$$(ii) \text{ and } c > 0 \quad : \quad t_0 \xrightarrow{w} N\left(-\frac{c}{2}\sqrt{\frac{\sigma_{2\varepsilon}^2}{\sigma_{4\varepsilon}} \frac{T\left(T-1\right)}{2}}, 1\right) \quad \text{as } N \to \infty \tag{24}$$

The limiting distribution of the normalized coefficient statistic $\bar{t}_0$ is given by

$$(i) \text{ and } c \geq 0 \quad : \quad \bar{t}_0 \xrightarrow{w} N\left(-c\sqrt{\frac{T\left(T-1\right)}{2}}, \frac{\sigma_{4\varepsilon}}{\sigma_{2\varepsilon}^2}\right) \quad \text{as } N \to \infty \tag{25}$$

$$(ii) \text{ and } c > 0 \quad : \quad \bar{t}_0 \xrightarrow{w} N\left(-\frac{c}{2}\sqrt{\frac{T\left(T-1\right)}{2}}, \frac{\sigma_{4\varepsilon}}{\sigma_{2\varepsilon}^2}\right) \quad \text{as } N \to \infty \tag{26}$$

The proof of Proposition 4 is given in Appendix A.3. The proposition shows that under the null hypothesis of a unit root, the t-statistic t_0 is asymptotically standard normal. So again unit root inference is carried out by employing critical values from the standard normal distribution. Further, the proposition shows that the local power of the tests is increasing in both T and $\sigma_{2\varepsilon}^2/\sigma_{4\varepsilon}$. Also, the local power of the tests is higher under the mean stationary alternative compared to under the covariance stationary alternative as the location parameter in the first case is twice as large in absolute value as in the second case. The difference is explained by the positive asymptotic bias of the Breitung-Meyer estimator $\hat{\rho}_0$ in the latter case. The test based on the normalized coefficient statistic $\bar{t}_0$ is asymptotically equivalent to the test based on the t-statistic t_0 when $\sigma_{4\varepsilon} = \sigma_{2\varepsilon}^2$. When this is not the case, the test based on the normalized coefficient statistic will be distorted when employing critical values from the standard normal distribution. In a one-sided test it will tend to reject the null hypothesis of a unit root too often when $\sigma_{4\varepsilon} > \sigma_{2\varepsilon}^2$. In this case the test is oversized. The opposite is true when $\sigma_{4\varepsilon} < \sigma_{2\varepsilon}^2$. So unless there is any prior knowledge about the ratio $\sigma_{2\varepsilon}^2/\sigma_{4\varepsilon}$, the unit root test should be based on the t-statistic. Note, that if $\sigma_{2\varepsilon}^2 \neq \sigma_{4\varepsilon}$ such that there is a difference between the local power of the two tests, this difference decreases as T increases. However, the size distortion is not affected by T and hence it remains as T increases.

The advantage of using the Breitung-Meyer unit root test is that the local power only depends on one nuisance parameter. Further, the test is invariant with respect to the individual-specific levels even in finite samples. This means that the size of the test is invariant with respect to the initial values and the power of the test is invariant with respect to the individual-specific term α_i under mean stationary alternatives.

3.3 Harris-Tzavalis

The within-group transformation of the original model is obtained by subtracting the individual time-series means from the variables in equation (1). This yields the following regression model

$$\begin{gathered} y_{it} - \tfrac{1}{T}\sum_{t=1}^{T} y_{it} = \rho\left(y_{it-1} - \tfrac{1}{T}\sum_{t=1}^{T} y_{it-1}\right) + w_{it} \\ w_{it} = \varepsilon_{it} - \tfrac{1}{T}\sum_{t=1}^{T}\varepsilon_{it} \end{gathered} \qquad \text{for } i = 1, ..., N \text{ and } t = 1, ..., T \tag{27}$$

The within-group estimator of ρ is then defined by

$$\hat{\rho}_{WG} = \left(\sum_{i=1}^{N} y'_{i,-1} Q_T y_{i,-1}\right)^{-1} \left(\sum_{i=1}^{N} y'_{i,-1} Q_T y_i\right) \tag{28}$$

where Q_T is a $T \times T$ symmetric and idempotent matrix defined as $Q_T = I_T - \frac{1}{T}\iota_T\iota'_T$ where I_T is the $T \times T$ identity matrix and $\iota_T\iota'_T$ is a $T \times T$ matrix of ones. It is well-known that this estimator is inconsistent when $|\rho| < 1$. The asymptotic bias is often referred to as the Nickell-bias since Nickell (1981) is the first to provide an analytical expression for it. Under the assumption about the time-series processes being covariance stationary, the asymptotic bias is a function of ρ and T which is always negative when $0 < \rho < 1$ and decreases numerically as T increases. Harris & Tzavalis (1999) show that the asymptotic bias of the within-group estimator equals $-3/(T+1)$ when $\rho = 1$. As this expression does not depend on any nuisance parameters, their idea is to base a unit root test on the bias adjusted within-group estimator. Proposition 5 below provides the limiting distribution of $\hat{\rho}_{WG}$ under both the null hypothesis of a unit root and the local alternative when ρ is local-to-unity.

Proposition 5 *Under Assumption 1, 2, 3, 4 and the local-to-unity sequence for* ρ *given by* $\rho = 1 - c/\sqrt{N}$*, the limiting distribution of the adjusted within-group estimator* $\hat{\rho}_{WG}$ *is given by*

$$(i) \text{ and } c \geq 0 \quad : \quad \sqrt{N}\left(\hat{\rho}_{WG} - \rho + \frac{3}{T+1}\right) \xrightarrow{w} N\left(-c\frac{T-2}{2(T+1)}, \frac{k_1 m_4 + k_2 \sigma_{4\varepsilon}}{\sigma_{2\varepsilon}^2}\right) \quad \text{as } N \to \infty \tag{29}$$

$$(ii) \text{ and } c > 0 \quad : \quad \sqrt{N}\left(\hat{\rho}_{WG} - \rho + \frac{3}{T+1}\right) \xrightarrow{w} N\left(c\frac{T+4}{4(T+1)}, \frac{k_1 m_4 + k_2 \sigma_{4\varepsilon}}{\sigma_{2\varepsilon}^2}\right) \quad \text{as } N \to \infty \tag{30}$$

where

$$k_1 = \frac{12(T-2)(2T-1)}{5T(T-1)(T+1)^3} \qquad k_2 = \frac{3\left(17T^3 - 44T^2 + 77T - 24\right)}{5T(T-1)(T+1)^3} \tag{31}$$

The proof of Proposition 5 is given in Appendix A.4. The proposition shows that except in the unit root case, the adjusted within-group estimator has an asymptotic bias of order $1/\sqrt{N}$ under the local alternative. The bias is negative under the assumption about mean stationarity and positive under the assumption about covariance stationarity. This means that the adjustment is respectively too big and too small. The limiting variance of $\hat{\rho}_{WG}$ is the same in the unit root case and under the two local alternatives. It depends on fourth order moments of the errors ε_{it} through the term m_4. As $k_1 < k_2$ the fourth order moments receive less weight than the squared second order moments.

Harris & Tzavalis (1999) assume that the errors ε_{it} are iid normally distributed across i such that $\sigma_{4\varepsilon} = \sigma_{2\varepsilon}^2$ and $m_4 = 3\sigma_{2\varepsilon}^2$. In this case, the limiting variance of $\hat{\rho}_{WG}$ only depends on T and is given by the following expression

$$\tilde{V}_{WG} = 3k_1 + k_2 = \frac{3\left(17T^2 - 20T + 17\right)}{5\left(T-1\right)\left(T+1\right)^3} \tag{32}$$

Therefore, Harris & Tzavalis (1999) suggest using the normalized coefficient statistic as a unit root test statistic. It is defined as follows

$$\bar{t}_{WG} = \tilde{V}_{WG}^{-\frac{1}{2}}\sqrt{N}\left(\hat{\rho}_{WG} - 1 + \frac{3}{T+1}\right) \tag{33}$$

However, as before it is also possible to use the usual t-statistic as a test statistic. White's heteroskedasticity consistent estimator of the limiting variance of the bias adjusted within-group estimator is given by the following expression

$$\hat{V}_{WG} = \left(\frac{1}{N}\sum_{i=1}^{N} y_{i,-1}' Q_T y_{i,-1}\right)^{-1} \frac{1}{N}\sum_{i=1}^{N} y_{i,-1}' Q_T \hat{\omega}_i \hat{\omega}_i' Q_T y_{i,-1} \left(\frac{1}{N}\sum_{i=1}^{N} y_{i,-1}' Q_T y_{i,-1}\right)^{-1} \tag{34}$$

where the vector of residuals is $\hat{w}_i = Q_T y_i - \hat{\rho}_{WG} Q_T y_{i,-1}$. The bias adjusted within-group t-statistic is then defined in the following way

$$t_{WG} = \hat{V}_{WG}^{-\frac{1}{2}}\sqrt{N}\left(\hat{\rho}_{WG} - 1 + \frac{3}{T+1}\right) \tag{35}$$

The limiting distributions of these test statistics are given in Proposition 6 below.

Proposition 6 *Under Assumption 1, 2, 3, 4 and the local-to-unity sequence for ρ given by $\rho = 1 - c/\sqrt{N}$, the limiting distribution of the adjusted within-group t-statistic t_{WG} is given by*

$$(i) \text{ and } c \geq 0 \quad : \quad t_{WG} \xrightarrow{w} N\left(-c\frac{3T}{2\left(T+1\right)}\sigma_{2\varepsilon}\sqrt{\left(k_1 m_4 + k_2\sigma_{4\varepsilon}\right)^{-1}}, 1\right) \quad \text{as } N \to \infty \tag{36}$$

$$(ii) \text{ and } c > 0 \quad : \quad t_{WG} \xrightarrow{w} N\left(-c\frac{3T}{4\left(T+1\right)}\sigma_{2\varepsilon}\sqrt{\left(k_1 m_4 + k_2\sigma_{4\varepsilon}\right)^{-1}}, 1\right) \quad \text{as } N \to \infty \tag{37}$$

The limiting distribution of the Harris-Tzavalis normalized coefficient statistic $\bar{t}_{WG}$ is given by

$$(i) \text{ and } c \geq 0 \quad : \quad \bar{t}_{WG} \xrightarrow{w} N\left(-c\frac{3T}{2\left(T+1\right)}\sqrt{\frac{5\left(T-1\right)\left(T+1\right)^3}{3\left(17T^2 - 20T + 17\right)}}, \frac{\tilde{k}_1 m_4 + \tilde{k}_2\sigma_{4\varepsilon}}{\sigma_{2\varepsilon}^2}\right) \quad \text{as } N \to \infty \tag{38}$$

$$(ii) \text{ and } c > 0 \quad : \quad \bar{t}_{WG} \xrightarrow{w} N\left(-c\frac{3T}{4\left(T+1\right)}\sqrt{\frac{5\left(T-1\right)\left(T+1\right)^3}{3\left(17T^2 - 20T + 17\right)}}, \frac{\tilde{k}_1 m_4 + \tilde{k}_2\sigma_{4\varepsilon}}{\sigma_{2\varepsilon}^2}\right) \quad \text{as } N \to \infty \tag{39}$$

where

$$\tilde{k}_1 = \frac{4\left(T-2\right)\left(2T-1\right)}{T\left(17T^2 - 20T + 17\right)} \qquad \tilde{k}_2 = \frac{17T^3 - 44T^2 + 77T - 24}{T\left(17T^2 - 20T + 17\right)} \tag{40}$$

The proof of Proposition 6 is given in Appendix A.4. Once again unit root inference based on the adjusted t-statistic t_{WG} can be carried out by employing critical values from the standard normal distribution. Further, the local power of the test statistics is increasing in T, $\sigma^2_{2\varepsilon}/\sigma_{4\varepsilon}$ and $\sigma^2_{2\varepsilon}/m_4$. It turns out that the local power is higher under the mean stationary alternative than under the covariance stationary alternative and we see that the location parameter in the first case is twice as large in absolute value as in the second case. This was also the case for the Breitung-Meyer test statistics, see Proposition 4. The unit root test based on the Harris-Tzavalis normalized coefficient statistic $\bar{t}_{WG}$ is asymptotically equivalent to test based on the t-statistic t_{WG} when the errors ε_{it} are normally distributed and homoskedastic across units. If at least one of these assumptions is violated, the test is likely to be distorted when employing critical values from the standard normal distribution. The test will tend to reject the null hypothesis too often when $\sigma_{4\varepsilon} > \sigma^2_{2\varepsilon}$ and when the excess kurtosis of ε_{it} is positive, i.e. $m_4 > 3\sigma^2_{2\varepsilon}$. Therefore, the Harris-Tzavalis normalized coefficient statistic should not be used for unit root inference unless the underlying assumptions have been verified by having been tested.

As with the Breitung-Meyer unit root test, the Harris-Tzavalis unit root test is invariant with respect to the individual-specific levels even in finite samples. However, the local power of the Harris-Tzavalis test depends on more nuisance parameters. A more serious disadvantage of this test is that the bias adjustment of the within-group estimator $\hat{\rho}_{WG}$ depends crucially on the errors ε_{it} being homoskedastic over time. If this assumption is violated the Harris-Tzavalis unit root test is likely to be distorted. To avoid this problem, Kruiniger & Tzavalis (2001) suggest using an estimator of the asymptotic bias in the adjustment of $\hat{\rho}_{WG}$. In the unit root case, the estimator of the asymptotic bias is consistent. However, in this paper we only investigate the performance of the unit tests when the errors ε_{it} are homoskedastic over time. Therefore, we do not consider this different bias adjustment in detail but we note that it is available.

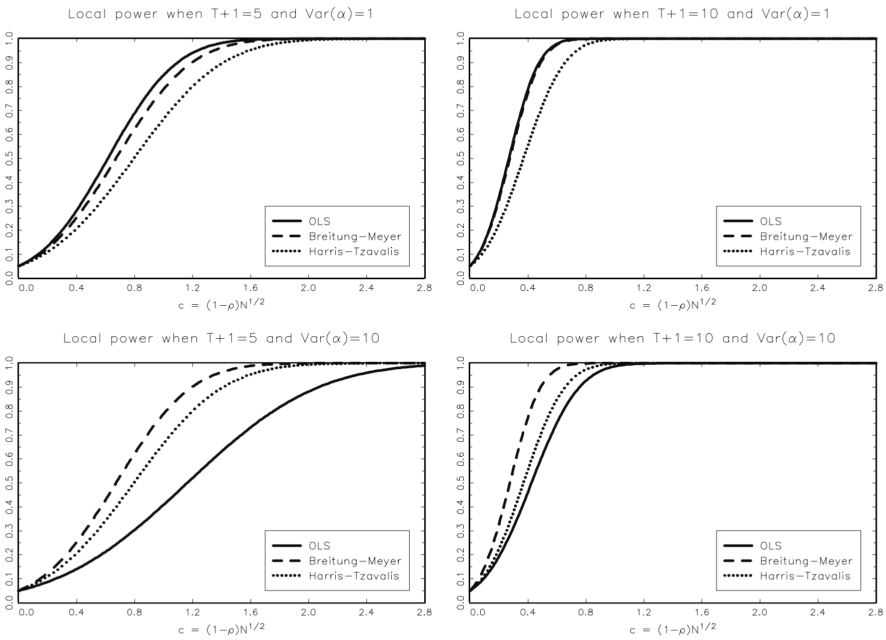
3.4 Comparison of the tests

Below we list the main findings about the local power of the tests based on the t-statistics. They follow immediately from the results in Proposition 2, 4 and 6.

1. The local power of the Breitung-Meyer test is always higher than the local power of the Harris-Tzavalis test.
2. Under the assumption about mean stationarity, the local power of the OLS test is higher than the local power of the Breitung-Meyer test when $\frac{\sigma^2_\alpha}{\sigma_{2\varepsilon}} < \frac{\sigma_{4\varepsilon}}{\sigma^2_{2\varepsilon}}\tau\left(1 + \frac{2\tau}{T-1}\right)$.
3. Under the assumption about covariance stationarity, the local power of the OLS test is higher than the local power of the Breitung-Meyer test when $\rho > \frac{T-5}{T-1}$.
4. Under the assumption about mean stationarity and when $\sigma_{4\varepsilon} = \sigma^2_{2\varepsilon}$ and $m_4 = 3\sigma^2_{2\varepsilon}$ the local power of the OLS test is higher than the local power of the Harris-Tzavalis test when $\frac{\sigma^2_\alpha}{\sigma_{2\varepsilon}} < \left(\frac{4(17T^2-20T+17)}{15T(T-1)(T+1)}\left(\tau + \frac{T-1}{2}\right) - 1\right)\left(\tau + \frac{T-1}{2}\right)$.

Figure 1 below illustrates some of these results. In each figure, the local power of one-sided tests at the 5% significance level based on the t-statistics is graphed as a function of $c = (1-\rho)\sqrt{N}$. As an example, when $\sigma_{4\varepsilon} = \sigma_{2\varepsilon}^2$ the local power of the Breitung-Meyer test is obtained as $\Phi\left(-1.645 + c\sqrt{T(T-1)/2}\right)$ where Φ denotes the distribution function of the standard normal. The local power is calculated under the assumption about mean stationarity and the following parameter values: $\tau = 1$, $\sigma_{4\varepsilon} = \sigma_{2\varepsilon}^2$ and $m_4 = 3\sigma_{2\varepsilon}^2$. The figures correspond to the value of $T+1$ being 5 or 10 and the value of $\sigma_\alpha^2/\sigma_{2\varepsilon}$ being 1 or 10. For this choice of parameters, the local power of the Breitung-Meyer test and the Harris-Tzavalis test only depends on T. We see that the local power of the Breitung-Meyer test is higher than the local power of the Harris-Tzavalis test for all values of c. When $\sigma_\alpha^2/\sigma_{2\varepsilon} = 1$ the local power of the OLS test is highest for all values of c, whereas when $\sigma_\alpha^2/\sigma_{2\varepsilon} = 10$ the local power of the OLS test is lowest for all values of c.

Figure 1: Comparison of the local power under mean stationarity



4 Simulation experiments

In this section the analytical results obtained in Section 3 are illustrated in a simulation experiment. The simulated model is the following

$$y_{i0} = 1_{\{|\rho|<1\}}\alpha_i + \varepsilon_{i0} \tag{41}$$

$$y_{it} = \rho y_{it-1} + (1-\rho)\alpha_i + \varepsilon_{it} \tag{42}$$

with

$$\varepsilon_{it} \sim \text{iid}N(0,1) \qquad \alpha_i \sim \text{iid}N\left(0,\sigma_\alpha^2\right) \qquad \varepsilon_{i0} \sim \text{iid}N(0,\tau(\rho)) \tag{43}$$

We consider different values of T, N and ρ which are $T+1 = 5, 10, 15$, $N = 100, 250, 500, 1000$ and $\rho = 0.90, 0.95, 0.99, 1.00$. The results are based on 5000 replications of the model. In Table 1 and 2 we report the empirical rejection probabilities of one-sided unit root tests based on the t-statistics with the critical value taken from the standard normal distribution at the nominal 5% significance level. For comparison the analytical rejection probabilities (i.e. the local power) are reported in brackets. We consider different simulation setups where the value of σ_α^2 is either 1 or 10. This parameter will only affect the OLS test as the two other tests do not depend on this parameter under the alternatives considered here. Further, the simulation setups depend on the variance of initial error term $\tau(\rho)$. Table 1 corresponds to the unit root case and the mean stationary alternative both with $\tau(\rho) = 1$ and Table 2 corresponds to the covariance stationary alternative with $\tau(\rho) = 1/\left(1-\rho^2\right)$.

In Table 1, we see that the empirical size of all tests is close to the nominal size of 0.05 and the empirical power is quite high even for values of ρ close to unity such as $\rho = 0.95$. Further, the increase in power can be quite dramatic when increasing $T+1$ from 5 to 10. For example, when $\rho = 0.99$ and $N = 1000$ the power of the Breitung-Meyer test increases from 0.20 to 0.57, the power of the Harris-Tzavalis test increases from 0.16 to 0.38, and the power of the OLS test increases from 0.21 to 0.59 when $\sigma_\alpha^2 = 1$ and from 0.12 to 0.33 when $\sigma_\alpha^2 = 10$. When comparing the different tests we see the results described in Section 3.4. To summarize, the power of the Breitung-Meyer test is always higher than the power of the Harris-Tzavalis test, and the OLS test has the highest (lowest) power of the three tests when $\sigma_\alpha^2 = 1$ ($\sigma_\alpha^2 = 10$). Finally, we see that the empirical rejection probabilities are quite close to the analytical rejection probabilities. This demonstrates that under the mean stationary alternative, the local power provides a good approximation to the actual power.

In Table 2, the most striking result is that the OLS test has very high power even for values of ρ very close to unity such as $\rho = 0.99$. According to the analytical results in Section 3.1, this will be the case unless the variability of the variable of interest is dominated by the variability of the individual-specific term. This is also the main conclusion from the simulation studies in the papers by Hall & Mairesse (2002) and Bond, Nauges & Windmeijer (2002) where the time-series processes are covariance stationary in the simulation setups. The empirical power of the OLS test is always higher than that of the Breitung-Meyer test and the Harris-Tzavalis test. In addition, the empirical power of the Breitung-Meyer test is always higher that of the Harris-Tzavalis test, and compared to Table 1 the empirical power of these tests is lower. These findings are all in accordance with the analytical results in Section 3. Again, we see that the empirical power is quite close to the analytical power except for the OLS test with $\sigma_\alpha^2 = 10$. As explained in Section 3.1, this is to be expected.

Finally, the tables in Appendix B contain more detailed information about the outcome from the simulation experiments. In addition, the appendix contains figures with a graphical comparison of the empirical and local power.

Table 1: Empirical and analytical (in brackets) rejection probabilities when $\tau(\rho) = 1$

ρ	$T+1$	N	OLS, $\sigma_\alpha^2 = 1$	OLS, $\sigma_\alpha^2 = 10$	Breitung-Meyer	Harris-Tzavalis
0.900	5	100	0.8032 (0.8480)	0.3702 (0.4088)	0.6838 (0.7895)	0.5624 (0.6665)
0.900	5	250	0.9892 (0.9951)	0.6660 (0.7228)	0.9566 (0.9871)	0.8654 (0.9491)
0.900	5	500	1.0000 (1.0000)	0.8962 (0.9354)	0.9992 (0.9999)	0.9906 (0.9986)
0.900	5	1000	1.0000 (1.0000)	0.9930 (0.9977)	1.0000 (1.0000)	1.0000 (1.0000)
0.900	10	100	1.0000 (1.0000)	0.9430 (0.9871)	0.9994 (1.0000)	0.9474 (0.9977)
0.900	10	250	1.0000 (1.0000)	0.9994 (1.0000)	1.0000 (1.0000)	0.9998 (1.0000)
0.900	10	500	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
0.900	10	1000	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
0.900	15	100	1.0000 (1.0000)	0.9998 (1.0000)	1.0000 (1.0000)	0.9978 (1.0000)
0.900	15	250	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
0.900	15	500	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
0.900	15	1000	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
0.950	5	100	0.3694 (0.3788)	0.1840 (0.1742)	0.3180 (0.3372)	0.2768 (0.2718)
0.950	5	250	0.6504 (0.6801)	0.2822 (0.2992)	0.5720 (0.6147)	0.4630 (0.4983)
0.950	5	500	0.8868 (0.9104)	0.4516 (0.4746)	0.8232 (0.8630)	0.6938 (0.7502)
0.950	5	1000	0.9916 (0.9951)	0.6828 (0.7228)	0.9748 (0.9871)	0.9240 (0.9491)
0.950	10	100	0.8796 (0.9218)	0.5674 (0.6147)	0.8548 (0.9123)	0.6200 (0.7231)
0.950	10	250	0.9978 (0.9993)	0.8864 (0.9218)	0.9966 (0.9990)	0.9102 (0.9708)
0.950	10	500	1.0000 (1.0000)	0.9888 (0.9964)	1.0000 (1.0000)	0.9950 (0.9996)
0.950	10	1000	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
0.950	15	100	0.9958 (0.9992)	0.8906 (0.9563)	0.9926 (0.9991)	0.8564 (0.9618)
0.950	15	250	1.0000 (1.0000)	0.9990 (0.9999)	1.0000 (1.0000)	0.9964 (0.9999)
0.950	15	500	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
0.950	15	1000	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
0.990	5	100	0.0968 (0.0842)	0.0780 (0.0664)	0.0936 (0.0808)	0.0924 (0.0753)
0.990	5	250	0.1146 (0.1108)	0.0810 (0.0776)	0.1108 (0.1043)	0.1034 (0.0940)
0.990	5	500	0.1502 (0.1475)	0.0908 (0.0920)	0.1452 (0.1363)	0.1276 (0.1188)
0.990	5	1000	0.2088 (0.2119)	0.1244 (0.1155)	0.1954 (0.1921)	0.1622 (0.1614)
0.990	10	100	0.1712 (0.1509)	0.1190 (0.1043)	0.1566 (0.1480)	0.1306 (0.1156)
0.990	10	250	0.2574 (0.2493)	0.1552 (0.1509)	0.2424 (0.2432)	0.1784 (0.1743)
0.990	10	500	0.3798 (0.3914)	0.2128 (0.2180)	0.3670 (0.3809)	0.2528 (0.2596)
0.990	10	1000	0.5926 (0.6147)	0.3292 (0.3372)	0.5672 (0.5997)	0.3822 (0.4090)
0.990	15	100	0.2528 (0.2475)	0.1750 (0.1650)	0.2526 (0.2448)	0.1766 (0.1682)
0.990	15	250	0.4340 (0.4511)	0.2746 (0.2795)	0.4258 (0.4457)	0.2728 (0.2863)
0.990	15	500	0.6770 (0.6941)	0.4168 (0.4424)	0.6646 (0.6873)	0.4182 (0.4535)
0.990	15	1000	0.9104 (0.9191)	0.6554 (0.6831)	0.9020 (0.9149)	0.6524 (0.6971)
1.000	5	100	0.0568 (0.0500)	0.0568 (0.0500)	0.0622 (0.0500)	0.0634 (0.0500)
1.000	5	250	0.0538 (0.0500)	0.0538 (0.0500)	0.0542 (0.0500)	0.0594 (0.0500)
1.000	5	500	0.0546 (0.0500)	0.0546 (0.0500)	0.0550 (0.0500)	0.0574 (0.0500)
1.000	5	1000	0.0520 (0.0500)	0.0520 (0.0500)	0.0504 (0.0500)	0.0552 (0.0500)
1.000	10	100	0.0638 (0.0500)	0.0638 (0.0500)	0.0624 (0.0500)	0.0640 (0.0500)
1.000	10	250	0.0564 (0.0500)	0.0564 (0.0500)	0.0600 (0.0500)	0.0608 (0.0500)
1.000	10	500	0.0548 (0.0500)	0.0548 (0.0500)	0.0486 (0.0500)	0.0558 (0.0500)
1.000	10	1000	0.0454 (0.0500)	0.0454 (0.0500)	0.0456 (0.0500)	0.0500 (0.0500)
1.000	15	100	0.0554 (0.0500)	0.0554 (0.0500)	0.0582 (0.0500)	0.0626 (0.0500)
1.000	15	250	0.0578 (0.0500)	0.0578 (0.0500)	0.0530 (0.0500)	0.0562 (0.0500)
1.000	15	500	0.0502 (0.0500)	0.0502 (0.0500)	0.0476 (0.0500)	0.0532 (0.0500)
1.000	15	1000	0.0442 (0.0500)	0.0442 (0.0500)	0.0458 (0.0500)	0.0490 (0.0500)

Table 2: Empirical and analytical (in brackets) rejection probabilities when $\tau(\rho) = 1/\left(1-\rho^2\right)$

ρ	$T+1$	N	OLS, $\sigma_\alpha^2 = 1$	OLS, $\sigma_\alpha^2 = 10$	Breitung-Meyer	Harris-Tzavalis
0.900	5	100	0.9966 (0.9977)	0.8826 (0.9977)	0.3358 (0.3372)	0.3014 (0.2718)
0.900	5	250	1.0000 (1.0000)	0.9972 (1.0000)	0.6036 (0.6147)	0.5192 (0.4983)
0.900	5	500	1.0000 (1.0000)	1.0000 (1.0000)	0.8512 (0.8630)	0.7600 (0.7502)
0.900	5	1000	1.0000 (1.0000)	1.0000 (1.0000)	0.9814 (0.9871)	0.9460 (0.9491)
0.900	10	100	1.0000 (1.0000)	0.9978 (1.0000)	0.8878 (0.9123)	0.7368 (0.7231)
0.900	10	250	1.0000 (1.0000)	1.0000 (1.0000)	0.9982 (0.9990)	0.9756 (0.9708)
0.900	10	500	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	0.9990 (0.9996)
0.900	10	1000	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
0.900	15	100	1.0000 (1.0000)	1.0000 (1.0000)	0.9966 (0.9991)	0.9652 (0.9618)
0.900	15	250	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (0.9999)
0.900	15	500	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
0.900	15	1000	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
0.950	5	100	0.9338 (0.9354)	0.7588 (0.9354)	0.1696 (0.1509)	0.1566 (0.1301)
0.950	5	250	0.9998 (0.9996)	0.9806 (0.9996)	0.2580 (0.2493)	0.2306 (0.2048)
0.950	5	500	1.0000 (1.0000)	1.0000 (1.0000)	0.3830 (0.3914)	0.3252 (0.3139)
0.950	5	1000	1.0000 (1.0000)	1.0000 (1.0000)	0.6032 (0.6147)	0.5038 (0.4983)
0.950	10	100	0.9996 (0.9990)	0.9788 (0.9990)	0.4314 (0.4424)	0.3202 (0.2993)
0.950	10	250	1.0000 (1.0000)	1.0000 (1.0000)	0.7586 (0.7663)	0.5592 (0.5492)
0.950	10	500	1.0000 (1.0000)	1.0000 (1.0000)	0.9474 (0.9563)	0.8086 (0.8040)
0.950	10	1000	1.0000 (1.0000)	1.0000 (1.0000)	0.9984 (0.9990)	0.9722 (0.9708)
0.950	15	100	1.0000 (1.0000)	0.9994 (1.0000)	0.7438 (0.7703)	0.5560 (0.5254)
0.950	15	250	1.0000 (1.0000)	1.0000 (1.0000)	0.9762 (0.9832)	0.8670 (0.8546)
0.950	15	500	1.0000 (1.0000)	1.0000 (1.0000)	0.9998 (0.9999)	0.9862 (0.9852)
0.950	15	1000	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (0.9999)
0.990	5	100	0.4268 (0.4088)	0.3790 (0.4088)	0.0780 (0.0640)	0.0742 (0.0616)
0.990	5	250	0.7216 (0.7228)	0.6682 (0.7228)	0.0800 (0.0734)	0.0822 (0.0693)
0.990	5	500	0.9328 (0.9354)	0.8878 (0.9354)	0.0944 (0.0852)	0.0898 (0.0789)
0.990	5	1000	0.9982 (0.9977)	0.9922 (0.9977)	0.1108 (0.1043)	0.1000 (0.0940)
0.990	10	100	0.6946 (0.6831)	0.6324 (0.6831)	0.1006 (0.0893)	0.0956 (0.0776)
0.990	10	250	0.9566 (0.9563)	0.9308 (0.9563)	0.1298 (0.1209)	0.1048 (0.0983)
0.990	10	500	0.9984 (0.9990)	0.9960 (0.9990)	0.1618 (0.1650)	0.1298 (0.1262)
0.990	10	1000	1.0000 (1.0000)	1.0000 (1.0000)	0.2266 (0.2432)	0.1722 (0.1743)
0.990	15	100	0.8598 (0.8416)	0.8086 (0.8416)	0.1328 (0.1214)	0.1182 (0.0963)
0.990	15	250	0.9944 (0.9944)	0.9878 (0.9944)	0.1928 (0.1865)	0.1452 (0.1347)
0.990	15	500	1.0000 (1.0000)	1.0000 (1.0000)	0.2694 (0.2815)	0.1950 (0.1892)
0.990	15	1000	1.0000 (1.0000)	1.0000 (1.0000)	0.4260 (0.4457)	0.2782 (0.2863)

5 Conclusions

In this paper we have investigated the performance of some of the unit root tests which have been suggested in the literature. To do this we have derived the asymptotic power of the tests under local alternatives. We find that the local power of the Breitung-Meyer test is always higher than the local power of the Harris-Tzavalis test. In addition, the Harris-Tzavalis test is very sensitive to minor deviations from the underlying assumptions, such as the error terms being heteroskedastic instead of homoskedastic over time. Given these results, the Harris-Tzavalis unit root test seems useless. The results concerning the OLS test clearly demonstrate that the specification of the initial values is important. Under the covariance stationary alternative, the local power of the OLS test is substantially higher than the local power of the Breitung-Meyer test. Under the mean stationary alternative, this is less likely to be the case. The reason is that, unlike the Breitung-Meyer test, the OLS test is not invariant with respect to the individual-specific levels. Moreover, the local power of the OLS test is low when the variation in the individual-specific terms is high. Altogether, the Breitung-Meyer test seems to be more robust.

The tests considered in this paper are based on the assumption that the AR parameter is the same for all cross-section units. Nevertheless, it would be interesting to investigate the asymptotic power of the tests against local alternatives where the AR parameter differs across cross-section units. Since the corresponding LS estimators can be considered as estimators of the mean AR parameter, there are good reasons to expect the tests to have power against alternatives where the mean AR parameter is less than unity. For example the alternative where the AR parameter is less than unity for at least a group of cross-section units. However, the additional asymptotic bias of the LS estimators resulting from ignoring heterogeneity in the AR parameter will also affect the power. Other things being equal, a positive bias will have a negative effect on the power and vice versa. It would also be interesting to compare the unit root tests considered in this paper and the unit root test suggested by Im, Pesaran & Shin (2003) as they are based on very different test statistics. More specifically, the framework considered by Im, Pesaran & Shin (2003) explicitly allows for alternatives where the AR parameter is less than unity for at least a group of cross-section units and their test is based on the cross-section average of individual Dickey-Fuller test statistics. This suggests some directions for future research.

A Appendix

This appendix contains the proofs of the propositions in Section 3. The proofs are all based on standard asymptotic theory, see for example White (2001). We start out with some results that are useful in the following.

A.1 Preliminary lemmas and results

Lemma 1 *Under the local-to-unity sequence for ρ given by $\rho = 1 - c/N^k$ for $k, c > 0$ the following holds*

$$\rho^t = 1 - t\frac{c}{N^k} + o\left(N^{-k}\right) \tag{44}$$

$$\frac{1}{1-\rho^2} = \frac{N^k}{2c} + o\left(N^{-k}\right) \tag{45}$$

Proof: The binomial formula yields

$$\rho^t = \left(1 - \frac{c}{N^k}\right)^t = 1 - t\frac{c}{N^k} + \frac{t(t-1)}{2!}\frac{c^2}{N^{2k}} - \frac{t(t-1)(t-2)}{3!}\frac{c^3}{N^{3k}} + \ldots + \frac{(-c)^t}{N^{kt}}$$

and the results follow directly. $\square$

Lemma 2 *Let X and Y be random variables with $E|X|^r < \infty$ and $E|Y|^r < \infty$ for some $r > 0$. Then $E|X+Y|^r \leq c_r\left(E|X|^r + E|X|^r\right)$ where $c_r = 1$ for $r \leq 1$ and $c_r = 2^{r-1}$ for $r > 1$.*

Proof: See Proposition 3.8 in White (2001). $\square$

Lemma 3 *Let ε_i be a sequence of random variables with $E|\varepsilon_i|^{4+\delta} < K$ for some $\delta > 0$ and all $i = 1, ..., N$. Then for $k \leq 4$, $E|\varepsilon_i|^k < K + 1$ for all $i = 1, ..., N$.*

Proof: Using the inequality $|\varepsilon_i|^k \leq |\varepsilon_i|^m + 1$ for $k < m$ we have that $E|\varepsilon_i|^k \leq E|\varepsilon_i|^{4+\delta} + 1 < K + 1$ for $k \leq 4$. $\square$

In the following we consider different transformations of $\varepsilon_i = (\varepsilon_{i1}, ..., \varepsilon_{it})'$. We use that ι_T is a $T \times 1$ vector of ones and that $C_T(\rho)$ is the $T \times T$ matrix and $A_T(\rho)$ is the $T \times 1$ vector defined as

$$C_T(\rho) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & \vdots & \vdots \\ \rho & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \rho^{T-2} & \cdots & \rho & 1 & 0 \end{bmatrix} \qquad A_T(\rho) = \begin{bmatrix} 1 \\ \rho \\ \rho^2 \\ \vdots \\ \rho^{T-1} \end{bmatrix} \tag{46}$$

Note that $C_T(\rho) = C_T(1) + O\left(N^{-k}\right)$ and $A_T(\rho) = \iota_T + O\left(N^{-k}\right)$ when $\rho = 1 - c/N^k$ according to Lemma 1.

Lemma 4 *Under Assumption 1, 4 and the local-to-unity sequence for ρ given by $\rho = 1 - c/N^k$ for $c \geq 0$ and $k > 0$ the following results hold*

$$E\left|\varepsilon_i' C_T(\rho)' C_T(\rho)\varepsilon_i\right|^{2+\delta_1} < K_1 < \infty \quad \text{for some } \delta_1 > 0 \quad \text{and all } i = 1, ..., N \tag{47}$$

$$E\left|\varepsilon_i' A_T(\rho) A_T(\rho)'\varepsilon_i\right|^{2+\delta_1} < K_2 < \infty \quad \text{for some } \delta_1 > 0 \quad \text{and all } i = 1, ..., N \tag{48}$$

$$E\left|\varepsilon_i' C_T(\rho)' \iota_T\iota_T' C_T(\rho)\varepsilon_i\right|^{2+\delta_1} < K_3 < \infty \quad \text{for some } \delta_1 > 0 \quad \text{and all } i = 1, ..., N \tag{49}$$

$$E\left|\varepsilon_i' C_T(\rho)'\varepsilon_i\right|^{2+\delta_1} < K_4 < \infty \quad \text{for some } \delta_1 > 0 \quad \text{and all } i = 1, ..., N \tag{50}$$

$$E\left|\varepsilon_i' C_T(\rho)' \iota_T\iota_T'\varepsilon_i\right|^{2+\delta_1} < K_5 < \infty \quad \text{for some } \delta_1 > 0 \quad \text{and all } i = 1, ..., N \tag{51}$$

Proof of Lemma 4:

To show (47) we use that

$$E\left|\sum_{s=1}^{t}\rho^{t-s}\varepsilon_{is}\right|^{4+\delta} \leq \left(\sum_{s=1}^{t}\left(E\left|\rho^{t-s}\varepsilon_{is}\right|^{4+\delta}\right)^{\frac{1}{4+\delta}}\right)^{4+\delta} \leq t^{4+\delta}\left(|1-c|^{T(4+\delta)}+1\right)K \tag{52}$$

where the first inequality follows by Minkowski's inequality and the second inequality follows from $|\rho|^{(t-s)(4+\delta)} = \left|1 - c/N^k\right|^{(t-s)(4+\delta)} \leq |1-c|^{T(4+\delta)} + 1$ together with $E\left|\varepsilon_{it}\right|^{4+\delta} < K$ which holds according to Assumption 4 (i). This implies that

$$\begin{aligned} E\left|\varepsilon_i' C_T(\rho)' C_T(\rho)\varepsilon_i\right|^{2+\delta/2} &= E\left|\sum_{t=1}^{T-1}\left(\sum_{s=1}^{t}\rho^{t-s}\varepsilon_{is}\right)^2\right|^{2+\delta/2} \leq \left(\sum_{t=1}^{T-1}\left(E\left|\sum_{s=1}^{t}\rho^{t-s}\varepsilon_{is}\right|^{4+\delta}\right)^{\frac{1}{2+\delta/2}}\right)^{2+\delta/2} \\ &\leq K\left(|1-c|^{T(4+\delta)}+1\right)\left(\sum_{t=1}^{T-1}t^2\right)^{2+\delta/2} \equiv K_1 < \infty \end{aligned}$$

where the first inequality follows by Minkowski's inequality and the second inequality follows by the result above. This proves (47). The results in (48) and (49) are shown in a similar manner.

To show (50) we use that

$$E\left|\sum_{s=1}^{t}\rho^{t-s}\varepsilon_{is}\varepsilon_{it}\right|^{2+\delta/2} \leq \left(E\left|\sum_{s=1}^{t}\rho^{t-s}\varepsilon_{is}\right|^{4+\delta} E\left|\varepsilon_{it}\right|^{4+\delta}\right)^{\frac{1}{2}} \leq t^{2+\delta/2}\left(|1-c|^{T(4+\delta)}+1\right)^{\frac{1}{2}}K$$

where the first inequality follows by the Cauchy-Schwarz inequality and the second inequality follows from (52) above and $E\left|\varepsilon_{it}\right|^{4+\delta} < K$. This implies that

$$\begin{aligned} E\left|\varepsilon_i' C_T(\rho)'\varepsilon_i\right|^{2+\delta/2} &= E\left|\sum_{t=1}^{T}\sum_{s=1}^{t}\rho^{t-s}\varepsilon_{is}\varepsilon_{it}\right|^{2+\delta/2} \leq \left(\sum_{t=1}^{T}\left(E\left|\sum_{s=1}^{t}\rho^{t-s}\varepsilon_{is}\varepsilon_{it}\right|^{2+\delta/2}\right)^{\frac{1}{2+\delta/2}}\right)^{2+\delta/2} \\ &\leq K\left(|1-c|^{T(4+\delta)}+1\right)^{\frac{1}{2}}\left(\sum_{t=1}^{T}t\right)^{2+\delta/2} \equiv K_4 < \infty \end{aligned}$$

where the first inequality follows by Minkowski's inequality and the second inequality follows by the result above. This proves (50). The result in (51) is shown in a similar manner. $\square$

Lemma 5 *Under Assumption 1, 4 and the local-to-unity sequence for ρ given by $\rho = 1 - c/N^k$ for $c \geq 0$ and $k > 0$ the following results hold*

$$\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i' C_T(\rho)' C_T(\rho)\varepsilon_i \xrightarrow{P} \sigma_{2\varepsilon}\frac{T(T-1)}{2} \quad \text{as } N \to \infty \tag{53}$$

$$\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i' C_T(\rho)' \iota_T \iota_T' C_T(\rho)\varepsilon_i \xrightarrow{P} \sigma_{2\varepsilon}\frac{T(T-1)(2T-1)}{6} \quad \text{as } N \to \infty \tag{54}$$

Proof of Lemma 5:

To show (53) we use Markov's Law of Large Numbers which can be applied according to (47) in Lemma 4. This gives the following result

$$\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i' C_T(\rho)' C_T(\rho)\varepsilon_i - \frac{1}{N}\sum_{i=1}^{N}E\left(\varepsilon_i' C_T(\rho)' C_T(\rho)\varepsilon_i\right) \xrightarrow{P} 0 \quad \text{as } N \to \infty$$

We also have that

$$\frac{1}{N}\sum_{i=1}^{N}E\left(\varepsilon_i' C_T(\rho)' C_T(\rho)\varepsilon_i\right) = \frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^2 \operatorname{tr}\left(C_T(\rho)' C_T(\rho)\right) \to \sigma_{2\varepsilon}\frac{T(T-1)}{2} \quad \text{as } N \to \infty$$

as $\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^2 \to \sigma_{2\varepsilon}$ as $N \to \infty$ by Assumption 4 (ii), $\operatorname{tr}\left(C_T(\rho)' C_T(\rho)\right) = \operatorname{tr}\left(C_T(1)' C_T(1)\right) + O\left(N^{-k}\right)$ by Lemma 1 and $\operatorname{tr}\left(C_T(1)' C_T(1)\right) = \frac{T(T-1)}{2}$. Altogether this proves the result in (53).

By using similar arguments we show (54). In this case Markov's Law of Large Numbers can be applied according to (49) in Lemma 4. We also have that

$$\frac{1}{N}\sum_{i=1}^{N}E\left(\varepsilon_i' C_T(\rho)' \iota_T \iota_T' C_T(\rho)\varepsilon_i\right) = \frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^2 \iota_T' C_T(\rho) C_T(\rho)' \iota_T \to \sigma_{2\varepsilon}\frac{T(T-1)(2T-1)}{6} \quad N \to \infty$$

as $\iota_T' C_T(\rho) C_T(\rho)' \iota_T = \iota_T' C_T(1) C_T(1)' \iota_T + O\left(N^{-k}\right) = \frac{T(T-1)(2T-1)}{6} + O\left(N^{-k}\right)$. This proves the result in (54). □

Lemma 6 *Under Assumption 1 the following result hold*

$$E\left(\left(\varepsilon_i' C_T(\rho)' \varepsilon_i\right)^2\right) = \sigma_{i\varepsilon}^4 \operatorname{tr}\left(C_T(\rho)' C_T(\rho)\right) \tag{55}$$

Proof of Lemma 6:

We have that

$$\begin{aligned}
E\left(\varepsilon_i' C_T(\rho)' \varepsilon_i\right)^2 &= E\left(\sum_{t=2}^{T}\left(\rho^{t-2}\varepsilon_{i1} + \ldots + \varepsilon_{it-1}\right)\varepsilon_{it}\sum_{s=2}^{T}\left(\rho^{s-2}\varepsilon_{i1} + \ldots + \varepsilon_{is-1}\right)\varepsilon_{is}\right) \\
&= \sum_{t=2}^{T}\sum_{s=2}^{T}E\left(\left(\rho^{t-2}\varepsilon_{i1} + \ldots + \varepsilon_{it-1}\right)\left(\rho^{s-2}\varepsilon_{i1} + \ldots + \varepsilon_{is-1}\right)\varepsilon_{it}\varepsilon_{is}\right) \\
&= \sum_{t=2}^{T}E\left(\left(\rho^{t-2}\varepsilon_{i1} + \ldots + \varepsilon_{it-1}\right)^2 \varepsilon_{it}^2\right) = \sum_{t=2}^{T}E\left(\left(\rho^{t-2}\varepsilon_{i1} + \ldots + \varepsilon_{it-1}\right)^2\right)E\left(\varepsilon_{it}^2\right) \\
&= \sigma_{i\varepsilon}^4\sum_{t=2}^{T}\left(\rho^{2(t-2)} + \ldots + \rho^2 + 1\right) \\
&= \sigma_{i\varepsilon}^4 \operatorname{tr}\left(C_T(\rho)' C_T(\rho)\right)
\end{aligned}$$

where the third and fourth line hold since ε_{it} and ε_{is} for $s \neq t$ are independent with means zero implying that $\left(\rho^{t-2}\varepsilon_{i1} + ... + \varepsilon_{it-1}\right)$ and ε_{is} are independent when $t \leq s$ with $E\left(\left(\rho^{t-2}\varepsilon_{i1} + ... + \varepsilon_{it-1}\right)^2\right) = \sigma^2_{i\varepsilon}\left(\rho^{2(t-2)} + ... + \rho^2 + 1\right)$. $\square$

A.2 Proofs of the propositions in Section 3.1: OLS

For $-1 < \rho \leq 1$ the following expression for y_{it} is obtained by recursive substitution in (1)

$$y_{it} = \left(1-\rho^t\right)\alpha_i + \rho^t y_{i0} + \rho^{t-1}\varepsilon_{i1} + ... + \varepsilon_{it} \quad \text{for } t = 1, ..., T \tag{56}$$

Inserting the expression for the initial value given in Assumption 3 yields

$$y_{it} = \mathbf{1}_{\{|\rho|<1\}}\alpha_i + \rho^t\sqrt{\tau(\rho)}\varepsilon_{i0} + \rho^{t-1}\varepsilon_{i1} + ... + \varepsilon_{it} \quad \text{for } t = 0, ..., T \tag{57}$$

Using stacked notation, the regressor $y_{i,-1}$ and the regression error u_i can be expressed as follows

$$\begin{aligned} y_{i,-1} &= \mathbf{1}_{\{|\rho|<1\}}\alpha_i\iota_T + C_T(\rho)\varepsilon_i + A_T(\rho)\sqrt{\tau(\rho)}\varepsilon_{i0} && (58)\\ u_i &= (1-\rho)\alpha_i\iota_T + \varepsilon_i && (59) \end{aligned}$$

where ι_T is a $T \times 1$ vector of ones and $C_T(\rho)$ is the $T \times T$ matrix and $A_T(\rho)$ is the $T \times 1$ vector defined in (46).

Proof of Proposition 1:

Using the equation in (9) we have that

$$N^k\left(\hat{\rho}_{OLS} - \rho\right) = \left(\frac{1}{N^{2k}}\sum_{i=1}^N y'_{i,-1}y_{i,-1}\right)^{-1}\frac{1}{N^k}\sum_{i=1}^N y'_{i,-1}u_i \quad \text{for } k > 0 \tag{60}$$

Proposition 1 now follows by the results in Lemma 7 below.

Lemma 7 *Under Assumption 1, 2, 3(i), 4 and the local-to-unity sequence for ρ given by $\rho = 1 - c/\sqrt{N}$ for $c \geq 0$, the following results hold*

$$\frac{1}{N}\sum_{i=1}^N y'_{i,-1}y_{i,-1} \xrightarrow{P} T\left(\mathbf{1}_{\{c>0\}}\sigma^2_\alpha + \left(\tau + \frac{T-1}{2}\right)\sigma_{2\varepsilon}\right) \quad \textit{as } N \to \infty \tag{a1}$$

$$\frac{1}{\sqrt{N}}\sum_{i=1}^N y'_{i,-1}u_i \xrightarrow{w} N\left(cT\sigma^2_\alpha, T\left(\mathbf{1}_{\{c>0\}}\sigma^2_\alpha\sigma_{2\varepsilon} + \left(\tau + \frac{T-1}{2}\right)\sigma_{4\varepsilon}\right)\right) \quad \textit{as } N \to \infty \tag{b1}$$

Under Assumption 1, 2, 3(ii), 4 and the local-to-unity sequence for ρ given by $\rho = 1 - \tilde{c}/N$ for $\tilde{c} > 0$, the following results hold

$$\frac{1}{N^2}\sum_{i=1}^N y'_{i,-1}y_{i,-1} \xrightarrow{P} \frac{1}{2\tilde{c}}T\sigma_{2\varepsilon} \quad \textit{as } N \to \infty \tag{a2}$$

$$\frac{1}{N}\sum_{i=1}^N y'_{i,-1}u_i \xrightarrow{w} N\left(0, \frac{1}{2\tilde{c}}T\sigma_{4\varepsilon}\right) \quad \textit{as } N \to \infty \tag{b2}$$

Proof of Lemma 7:

(a1) Using the expression for $y_{i,-1}$ given in equation (58) we have

$$y_{i,-1}^{\prime} y_{i,-1} = R_{1i} + 2R_{2i} + 2R_{3i} + 2R_{4i} \tag{61}$$

where

$$\begin{aligned} R_{1i} &= \mathbf{1}_{\{|\rho|<1\}} \alpha_i^2 \iota_T^{\prime} \iota_T + \varepsilon_i^{\prime} C_T(\rho)^{\prime} C_T(\rho) \varepsilon_i + \varepsilon_{i0}^2 \tau(\rho) A_T(\rho)^{\prime} A_T(\rho) \\ R_{2i} &= \mathbf{1}_{\{|\rho|<1\}} \alpha_i \iota_T^{\prime} C_T(\rho) \varepsilon_i \\ R_{3i} &= \mathbf{1}_{\{|\rho|<1\}} \alpha_i \iota_T^{\prime} A_T(\rho) \sqrt{\tau(\rho)} \varepsilon_{i0} \\ R_{4i} &= \varepsilon_i^{\prime} C_T(\rho)^{\prime} A_T(\rho) \sqrt{\tau(\rho)} \varepsilon_{i0} \end{aligned}$$

We prove the result in (a1) by showing that

$$\frac{1}{N} \sum_{i=1}^{N} R_{1i} \xrightarrow{P} T\left(\mathbf{1}_{\{c>0\}} \sigma_\alpha^2 + \left(\tau + \frac{T-1}{2}\right) \sigma_{2\varepsilon}\right) \quad \text{as } N \to \infty \tag{62}$$

$$\frac{1}{N} \sum_{i=1}^{N} R_{ki} \xrightarrow{P} 0 \quad \text{as } N \to \infty \quad \text{for } k = 2,3,4 \tag{63}$$

The result in (62) is obtained by using the following

$$\frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\{|\rho|<1\}} \alpha_i^2 \iota_T^{\prime} \iota_T \xrightarrow{P} T \mathbf{1}_{\{c>0\}} \sigma_\alpha^2 \quad \text{as } N \quad \to \quad \infty \tag{64}$$

$$\frac{1}{N} \sum_{i=1}^{N} \varepsilon_i^{\prime} C_T(\rho)^{\prime} C_T(\rho) \varepsilon_i \xrightarrow{P} \sigma_{2\varepsilon} \frac{T(T-1)}{2} \quad \text{as } N \quad \to \quad \infty \tag{65}$$

$$\frac{1}{N} \sum_{i=1}^{N} \varepsilon_{i0}^2 \tau(\rho) A_T(\rho)^{\prime} A_T(\rho) \xrightarrow{P} T \sigma_{2\varepsilon} \tau \quad \text{as } N \quad \to \quad \infty \tag{66}$$

The first holds since $\frac{1}{N} \sum_{i=1}^{N} \alpha_i^2 \xrightarrow{P} \sigma_\alpha^2$ by Kolmogorov's Law of Large Numbers. The second result holds according to (53) in Lemma 5. The third result holds since $\frac{1}{N} \sum_{i=1}^{N} \varepsilon_{i0}^2 \xrightarrow{P} \sigma_{2\varepsilon}$ as $N \to \infty$ by Markov's Large of Large Numbers which can be applied since $E\left|\varepsilon_{i0}\right|^{4+\delta} < K$ for all $i = 1, ..., N$ in combination with that $\tau(\rho) = \tau$ and $A_T(\rho) = \iota_T + O\left(N^{-\frac{1}{2}}\right)$ such that $A_T(\rho)^{\prime} A_T(\rho) = \iota_T^{\prime} \iota_T + O\left(N^{-\frac{1}{2}}\right) = T + O\left(N^{-\frac{1}{2}}\right)$.

Next, we show (63) by showing that $E\left|\frac{1}{N} \sum_{i=1}^{N} R_{ki}\right|^2 \to 0$ as $N \to \infty$ for $k = 2,3,4$. We have that

$$E\left|\frac{1}{N} \sum_{i=1}^{N} R_{ki}\right|^2 = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E\left(R_{ki} R_{kj}\right) = \frac{1}{N^2} \sum_{i=1}^{N} E\left(R_{ki}^2\right) \quad \text{for } k = 2,3,4$$

as the sequence R_{ki} is independent across i with $E\left(R_{ki}\right) = 0$ as α_i, ε_{i0} and ε_i are all independent of each other with means zero. This gives

$$\frac{1}{N^2} \sum_{i=1}^{N} E\left(R_{2i}^2\right) = \frac{1}{N} \frac{1}{N} \sum_{i=1}^{N} \sigma_{i\varepsilon}^2 \mathbf{1}_{\{|\rho|<1\}} \sigma_\alpha^2 \iota_T^{\prime} C_T(\rho) C_T(\rho)^{\prime} \iota_T = \frac{1}{N} O(1)$$

which holds as $\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{2}=O(1)$ and $\iota_T' C_T(\rho) C_T(\rho)' \iota_T = O(1)$. This shows that $E\left|\frac{1}{N}\sum_{i=1}^{N} R_{2i}\right|^2 \to 0$ as $N\to\infty$. We also have

$$\frac{1}{N^2}\sum_{i=1}^{N} E\left(R_{3i}^2\right) = \frac{1}{N}\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{2}\mathbf{1}_{\{|\rho|<1\}}\sigma_\alpha^2 \iota_T' A_T(\rho) A_T(\rho)' \iota_T \tau = \frac{1}{N}O(1)$$

which holds as $\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{2}=O(1)$ and $\iota_T' A_T(\rho) A_T(\rho)' \iota_T = O(1)$. This shows that $E\left|\frac{1}{N}\sum_{i=1}^{N} R_{3i}\right|^2 \to 0$ as $N\to\infty$. Finally, by using similar argument we have

$$\frac{1}{N^2}\sum_{i=1}^{N} E\left(R_{4i}^2\right) = \frac{1}{N}\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{4} A_T(\rho)' C_T(\rho) C_T(\rho)' A_T(\rho)\tau = \frac{1}{N}O(1)$$

This shows that $E\left|\frac{1}{N}\sum_{i=1}^{N} R_{4i}\right|^2 \to 0$ as $N\to\infty$. Altogether, we have obtained the desired limits and (a1) is proved.

(b1) Using the expression for $y_{i,-1}$ given in (58) and the expression for u_i given in (59) we have

$$y_{i,-1}' u_i = Q_{1i} + Q_{2i} + Q_{3i} + Q_{4i} \tag{67}$$

where

$$\begin{aligned}
Q_{1i} &= \mathbf{1}_{\{|\rho|<1\}}\alpha_i \varepsilon_i' \iota_T + \varepsilon_i' C_T(\rho)' \varepsilon_i + \varepsilon_i' A_T(\rho)\sqrt{\tau(\rho)}\varepsilon_{i0} \\
Q_{2i} &= \alpha_i^2 (1-\rho) T \\
Q_{3i} &= \alpha_i (1-\rho)\iota_T' C_T(\rho)\varepsilon_i \\
Q_{4i} &= \alpha_i (1-\rho)\iota_T' A_T(\rho)\sqrt{\tau(\rho)}\varepsilon_{i0}
\end{aligned}$$

We prove the result in (b1) by showing that

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(Q_{1i}+E(Q_{2i})\right) \xrightarrow{w} N\left(cT\sigma_\alpha^2, T\left(\mathbf{1}_{\{c>0\}}\sigma_\alpha^2\sigma_{2\varepsilon} + \left(\tau + \frac{T-1}{2}\right)\sigma_{4\varepsilon}\right)\right) \quad \text{as } N\to\infty \tag{68}$$

and

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(Q_{2i}-E(Q_{2i})\right) \xrightarrow{P} 0 \quad \text{as } N\to\infty \tag{69}$$

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N} Q_{ki} \xrightarrow{P} 0 \quad \text{as } N\to\infty, \quad \text{for } k=3,4 \tag{70}$$

To show the result in (68) we use that

$$E(Q_{1i}) + E(Q_{2i}) = E(Q_{2i}) = \mathbf{1}_{\{|\rho|<1\}}\sigma_\alpha^2(1-\rho)T \tag{71}$$

$$\begin{aligned}
\operatorname{Var}\left(Q_{1i}+E(Q_{2i})\right) &= E\left(Q_{1i}^2\right) \\
&= \mathbf{1}_{\{|\rho|<1\}}\sigma_\alpha^2\sigma_{i\varepsilon}^2 T + \sigma_{i\varepsilon}^4 \operatorname{tr}\left(C_T(\rho)' C_T(\rho)\right) + \sigma_{i\varepsilon}^4 A_T(\rho)' A_T(\rho)\tau(\rho)
\end{aligned} \tag{72}$$

where we have used that ε_{i0}, α_i and ε_i are all independent of each other with means zero such that all covariances between the terms in Q_{1i} are zero. In addition we have used the result in Lemma 6. Using this we have

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}E\left(Q_{1i}+Q_{2i}\right) \rightarrow cT\sigma_{\alpha}^{2} \quad \text{as } N\rightarrow\infty$$

$$\frac{1}{N}\sum_{i=1}^{N}\operatorname{Var}\left(Q_{1i}+E\left(Q_{2i}\right)\right) \rightarrow \mathbf{1}_{\{c>0\}}T\sigma_{\alpha}^{2}\sigma_{2\varepsilon}+T\left(\tau+\frac{T-1}{2}\right)\sigma_{4\varepsilon}^{2} \quad \text{as } N\rightarrow\infty$$

which holds as $\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{2}\rightarrow\sigma_{2\varepsilon}$, $\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{4}\rightarrow\sigma_{4\varepsilon}$ as $N\rightarrow\infty$ and $\operatorname{tr}\left(C_{T}\left(\rho\right)'C_{T}\left(\rho\right)\right)=\frac{T(T-1)}{2}+O\left(N^{-\frac{1}{2}}\right)$ and $A_{T}\left(\rho\right)'A_{T}\left(\rho\right)=T+O\left(N^{-\frac{1}{2}}\right)$ by Lemma 1. The result in (68) now follows by the Liapounov Central Limit Theorem which can be applied since $E\left|Q_{1i}\right|^{2+\delta'}<K'$ for some $\delta'>0$ and all $i=1,...,N$. Letting $\delta'=\min\{2,\delta_{1}\}$ this is seen in the following way

$$\begin{aligned} E\left|Q_{1i}\right|^{2+\delta'} &\leq 2^{2+2\delta'}\left(E\left|\alpha_{i}\right|^{2+\delta'}E\left|\varepsilon_{i}'\iota_{T}\right|^{2+\delta'}+E\left|\varepsilon_{i}'C_{T}\left(\rho\right)'\varepsilon_{i}\right|^{2+\delta'}+E\left|\varepsilon_{i}'A_{T}\left(\rho\right)\right|^{2+\delta\prime}E\left|\tau\varepsilon_{i0}\right|^{2+\delta'}\right) \\ &\leq 2^{2+2\delta'}\left(\left(E\left(\alpha_{i}^{4}\right)+1\right)\left(K_{2}+1\right)+K_{4}+\tau^{2+\delta'}\left(K+1\right)\left(K_{2}+1\right)\right) \end{aligned} \tag{73}$$

where the first inequality follows by Lemma 2 and by using that α_{i} and ε_{i0} are both independent of ε_{i} and the second inequality follows by the result in Lemma 3 together with (48) and (50) in Lemma 4. Here we have used that $\varepsilon_{i}'\iota_{T}=\varepsilon_{i}'A_{T}\left(1\right)$.

Next, to show (69) we show that $E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(Q_{2i}-E\left(Q_{2i}\right)\right)\right|\rightarrow0$ as $N\rightarrow\infty$. We have that

$$\begin{aligned} E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(Q_{2i}-E\left(Q_{2i}\right)\right)\right|^{2} &= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\operatorname{Cov}\left(Q_{2i},Q_{2j}\right)=\frac{1}{N}\sum_{i=1}^{N}\operatorname{Var}\left(Q_{2i}\right) \\ &\leq \frac{1}{N}\sum_{i=1}^{N}E\left(\alpha_{i}^{4}\right)\left(1-\rho\right)^{2}T^{2}=O\left(N^{-1}\right) \end{aligned}$$

where the first line holds as Q_{2i} is independent across i, the first inequality holds as $\operatorname{Var}\left(Q_{2i}\right)\leq E\left(Q_{2i}^{2}\right)$ and the last equality sign holds as $E\left(\alpha_{i}^{4}\right)=O\left(1\right)$ and $\left(1-\rho\right)^{2}=O\left(N^{-1}\right)$. To show (70) we show that $E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Q_{ki}\right|\rightarrow0$ as $N\rightarrow\infty$ for $k=3,4$. We have that

$$\begin{aligned} E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Q_{3i}\right|^{2} &= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}E\left(Q_{3i}Q_{3j}\right)=\frac{1}{N}\sum_{i=1}^{N}E\left(Q_{3i}^{2}\right) \\ &= \frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{2}\sigma_{\alpha}^{2}\left(1-\rho\right)^{2}\iota_{T}'C_{T}\left(\rho\right)C_{T}\left(\rho\right)'\iota_{T}=O\left(N^{-1}\right) \end{aligned}$$

where first line holds as Q_{3i} is independent across i with mean zero since α_{i} and ε_{i} are independent with means zero, and the last line holds as $\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{2}=O\left(1\right)$, $\iota_{T}'C_{T}\left(\rho\right)C_{T}\left(\rho\right)'\iota_{T}=O\left(1\right)$ and $\left(1-\rho\right)^{2}=O\left(N^{-1}\right)$. Thus, $E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Q_{3i}\right|^{2}\rightarrow0$ as $N\rightarrow\infty$. Finally, we also have that

$$\begin{aligned} E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Q_{4i}\right|^{2} &= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}E\left(Q_{4i}Q_{4j}\right)=\frac{1}{N}\sum_{i=1}^{N}E\left(Q_{4i}^{2}\right) \\ &= \frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{2}\sigma_{\alpha}^{2}\left(1-\rho\right)^{2}\iota_{T}'A_{T}\left(\rho\right)A_{T}\left(\rho\right)'\iota_{T}\tau=O\left(N^{-1}\right) \end{aligned}$$

where first line holds as Q_{4i} is independent across i with mean zero since α_i and ε_{i0} are independent with means zero, and the last line holds as $\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^2 = O(1)$, $\iota_T' A_T(\rho) A_T(\rho)' \iota_T = O(1)$ and $(1-\rho)^2 = O(N^{-1})$. Thus, $E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N} Q_{4i}\right|^2 \to 0$ as $N \to \infty$. Altogether, we have obtained the desired limits and the result in (b1) is proved.

(a2) Using the expression for $y_{i,-1}' y_{i,-1}$ given in (61) we prove the result by showing that

$$\frac{1}{N^2}\sum_{i=1}^{N} R_{1i} \xrightarrow{P} \frac{1}{2\tilde{c}} T\sigma_{2\varepsilon} \quad \text{as } N \to \infty \tag{74}$$

$$\frac{1}{N^2}\sum_{i=1}^{N} R_{ki} \xrightarrow{P} 0 \quad \text{as } N \to \infty \quad \text{for } k = 2,3,4 \tag{75}$$

As the results in (64) and (65) also hold for $\rho = 1 - \tilde{c}/N$ the probability limit of $\frac{1}{N^2}\sum_{i=1}^{N} R_{1i}$ is determined by the probability limit of the cross-section average of the third term in R_{1i}. We have that

$$\frac{1}{N^2}\sum_{i=1}^{N} \varepsilon_{i0}^2 \tau(\rho) A_T(\rho)' A_T(\rho) = \frac{1}{N}\frac{1}{N}\sum_{i=1}^{N} \varepsilon_{i0}^2 \frac{1}{1-\rho^2} A_T(\rho)' A_T(\rho) \xrightarrow{P} \frac{1}{2\tilde{c}} T\sigma_{2\varepsilon}^2 \quad \text{as } N \to \infty$$

as $\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{i0}^2 \xrightarrow{P} \sigma_{2\varepsilon}$ as $N \to \infty$, $A_T(\rho)' A_T(\rho) = T + O(N^{-1})$ and $\tau(\rho) = \frac{N}{2\tilde{c}} + o(N^{-1})$ by Lemma 1. This proves the result in (74). We show (75) by showing that $E\left|\frac{1}{N^2}\sum_{i=1}^{N} R_{ki}\right|^2 \to 0$ as $N \to \infty$ for $k = 2,3,4$. By using similar arguments as when showing (63) we have that

$$E\left|\frac{1}{N^2}\sum_{i=1}^{N} R_{ki}\right|^2 = \frac{1}{N^4}\sum_{i=1}^{N}\sum_{j=1}^{N} E(R_{ki}R_{kj}) = \frac{1}{N^4}\sum_{i=1}^{N} E(R_{ki}^2) \quad \text{for } k = 2,3,4$$

and that $\frac{1}{N^4}\sum_{i=1}^{N} E(R_{2i}^2) = \frac{1}{N^3}O(1)$, $\frac{1}{N^4}\sum_{i=1}^{N} E(R_{3i}^2) = \frac{1}{N^2}O(1)$ and $\frac{1}{N^4}\sum_{i=1}^{N} E(R_{4i}^2) = \frac{1}{N^2}O(1)$. This proves the result in (75) and (a2) is proved.

(b2) Using the expression for $y_{i,-1}' u_i$ given in (67) we prove the result by showing that

$$\frac{1}{N}\sum_{i=1}^{N}\left(Q_{1i} + E(Q_{2i})\right) \xrightarrow{w} N\left(0, \frac{1}{2\tilde{c}} T\sigma_{4\varepsilon}\right) \quad \text{as } N \to \infty \tag{76}$$

and

$$\frac{1}{N}\sum_{i=1}^{N}\left(Q_{2i} - E(Q_{2i})\right) \xrightarrow{P} 0 \quad \text{as } N \to \infty \tag{77}$$

$$\frac{1}{N}\sum_{i=1}^{N} Q_{ki} \xrightarrow{P} 0 \quad \text{as } N \to \infty, \quad \text{for } k = 3,4 \tag{78}$$

Markov's Law of Large Numbers the cross-section average of the first two terms in Q_{1i} converge to zero in probability when N tends to infinity. Hence the cross-section average of the third term in Q_{1i} determines the limiting distribution of $\frac{1}{N}\sum_{i=1}^{N} Q_{1i}$. We have that $E(\varepsilon_i' A_T(\rho)\varepsilon_{i0}) = 0$ as ε_i and ε_{i0} are independent and that

$$\frac{1}{N}\sum_{i=1}^{N} \mathrm{Var}(\varepsilon_i' A_T(\rho)\varepsilon_{i0}) = \frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^4 A_T(\rho)' A_T(\rho) \to T\sigma_{4\varepsilon} \quad \text{as } N \to \infty$$

Applying the Liapounov Central Limit Theorem yields

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\varepsilon_i' A_T(\rho)\varepsilon_{i0} \xrightarrow{w} N(0, T\sigma_{4\varepsilon}) \quad \text{as } N \to \infty$$

This in turn implies that

$$\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i' A_T(\rho)\varepsilon_{i0}\sqrt{\tau(\rho)} \xrightarrow{w} N\left(0, \frac{1}{2\tilde{c}}T\sigma_{4\varepsilon}\right) \quad asN \to \infty \tag{79}$$

as $\sqrt{\tau(\rho)/N} = \sqrt{1/(2\tilde{c})} + o\left(N^{-\frac{1}{2}}\right)$ by Lemma 1. Using that $\frac{1}{N}\sum_{i=1}^{N} E(Q_{2i}) = \frac{1}{N}\sum_{i=1}^{N}\sigma_\alpha^2 T\frac{\tilde{c}}{N} \to 0$ as $N \to \infty$ gives the result in (76). By using similar arguments as when showing (69) and (70) we have that

$$E\left|\frac{1}{N}\sum_{i=1}^{N} Q_{ki} - E(Q_{ki})\right|^2 \leq \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E(Q_{ki}Q_{kj}) = \frac{1}{N^2}\sum_{i=1}^{N} E\left(Q_{ki}^2\right) \quad \text{for } k = 2, 3, 4$$

and that $\frac{1}{N^2}\sum_{i=1}^{N} E\left(Q_{2i}^2\right) = \frac{1}{N}O\left(N^{-2}\right)$, $\frac{1}{N^2}\sum_{i=1}^{N} E\left(Q_{3i}^2\right) = \frac{1}{N}O\left(N^{-2}\right)$ and $\frac{1}{N^2}\sum_{i=1}^{N} E\left(Q_{4i}^2\right) = \frac{1}{N}O\left(N^{-1}\right)$. This proves the result in (76) and (77). Altogether, we have proved the result in (b2). □

Proof of Proposition 2:

Proposition 2 follows by the results already obtained and Lemma 8 given below.

Lemma 8 *Under Assumption 1, 2, 3(i), 4 and the local-to-unity sequence for ρ given by $\rho = 1 - c/\sqrt{N}$ for $c \geq 0$, the following result holds*

$$\frac{1}{N}\sum_{i=1}^{N} y_{i,-1}'\hat{u}_i\hat{u}_i' y_{i,-1} \xrightarrow{P} T\left(\mathbf{1}_{\{c>0\}}\sigma_\alpha^2\sigma_{2\varepsilon} + \left(\tau + \frac{T-1}{2}\right)\sigma_{4\varepsilon}\right) \quad \textit{as } N \to \infty \tag{a1}$$

Under Assumption 1, 2, 3(ii), 4 and the local-to-unity sequence for ρ given by $\rho = 1 - \tilde{c}/N$ for $\tilde{c} > 0$, the following result holds

$$\frac{1}{N^2}\sum_{i=1}^{N} y_{i,-1}'\hat{u}_i\hat{u}_i' y_{i,-1} \xrightarrow{P} \frac{1}{2\tilde{c}}T\sigma_{4\varepsilon} \quad \textit{as } N \to \infty \tag{a2}$$

Combining the results in (a1) and (a2) in Lemma 8 above with (a1) and (a2) in Lemma 7 we have that

$$\begin{aligned}
\hat{V}_{OLS}(k) &= \left(\frac{1}{N^{2k}}\sum_{i=1}^{N} y_{i,-1}' y_{i,-1}\right)^{-1} \frac{1}{N^{2k}}\sum_{i=1}^{N} y_{i,-1}'\hat{u}_i\hat{u}_i' y_{i,-1}\left(\frac{1}{N^{2k}}\sum_{i=1}^{N} y_{i,-1}' y_{i,-1}\right)^{-1} \\
&\xrightarrow{P} \begin{cases} \dfrac{1}{T}\dfrac{\mathbf{1}_{\{c>0\}}\sigma_\alpha^2\sigma_{2\varepsilon} + \left(\tau + \frac{T-1}{2}\right)\sigma_{4\varepsilon}}{\left(\mathbf{1}_{\{c>0\}}\sigma_\alpha^2 + \left(\tau + \frac{T-1}{2}\right)\sigma_{2\varepsilon}\right)^2} \quad \text{as } N \to \infty & \text{when } k = \frac{1}{2} \\ \tilde{c}\dfrac{\sigma_{4\varepsilon}}{\sigma_{2\varepsilon}^2}\dfrac{2}{T} \quad \text{as } N \to \infty & \text{when } k = 1 \end{cases}
\end{aligned}$$

Using the expression for t_{OLS} in (13) we have that

$$t_{OLS} = \hat{V}_{OLS}(k)^{-\frac{1}{2}} N^k(\hat{\rho}_{OLS} - 1) = \hat{V}_{OLS}(k)^{-\frac{1}{2}} N^k(\hat{\rho}_{OLS} - \rho) - c\hat{V}_{OLS}(k)^{-\frac{1}{2}} \tag{80}$$

Using this, the results in Proposition 1 yields the results in Proposition 2.

Proof of Lemma 8

Inserting the expression for $\hat{u}_i$ given by $\hat{u}_i = u_i + (\rho - \hat{\rho}_{OLS})\, y_{i,-1}$ yields

$$\begin{aligned}\frac{1}{N^{2k}}\sum_{i=1}^{N} y_{i,-1}'\hat{u}_i\hat{u}_i' y_{i,-1} &= \frac{1}{N^{2k}}\sum_{i=1}^{N} y_{i,-1}' u_i u_i' y_{i,-1} + (\rho - \hat{\rho}_{OLS})^2 \frac{1}{N^{2k}}\sum_{i=1}^{N} y_{i,-1}' y_{i,-1} y_{i,-1}' y_{i,-1} \\ &\quad + 2\,(\rho - \hat{\rho}_{OLS})\frac{1}{N^{2k}}\sum_{i=1}^{N} y_{i,-1}' y_{i,-1} y_{i,-1}' u_i\end{aligned}$$

As $N^k\,(\hat{\rho}_{OLS} - \rho) = O\,(1)$ by Proposition 1, we prove the result by showing that

$$(i) \quad : \quad \frac{1}{N}\sum_{i=1}^{N} y_{i,-1}' u_i u_i' y_{i,-1} \xrightarrow{P} T\left(\mathbf{1}_{\{c>0\}}\sigma_\alpha^2\sigma_{2\varepsilon} + \left(\tau + \frac{T-1}{2}\right)\sigma_{4\varepsilon}\right) \quad \text{as } N \to \infty \tag{81}$$

$$(ii) \quad : \quad \frac{1}{N^2}\sum_{i=1}^{N} y_{i,-1}' u_i u_i' y_{i,-1} \xrightarrow{P} \frac{1}{2\tilde{c}} T\sigma_{4\varepsilon} \quad \text{as } N \to \infty \tag{82}$$

$$\frac{1}{N^{4k}}\sum_{i=1}^{N} y_{i,-1}' y_{i,-1} y_{i,-1}' y_{i,-1} \xrightarrow{P} 0 \quad \text{as } N \to \infty \tag{83}$$

$$\frac{1}{N^{3k}}\sum_{i=1}^{N} y_{i,-1}' y_{i,-1} y_{i,-1}' u_i \xrightarrow{P} 0 \quad \text{as } N \to \infty \tag{84}$$

According to (61) we have $y_{i,-1}' y_{i,-1} = \sum_{k=1}^{4} R_{ki}$ and according to (67) we have $y_{i,-1}' u_i = \sum_{i=1}^{4} Q_{ki}$. Using these expressions and Lemma 3 we have

$$\begin{aligned}E\left(R_{1i}^2\right) &\leq 4\left(E\left(\alpha_i^4\right)T^2 + E\left(\left(\varepsilon_i' C_T(\rho)' C_T(\rho)\,\varepsilon_i\right)^2\right) + E\left(\varepsilon_{i0}^4\right)\tau(\rho)^2\left(A_T(\rho)' A_T(\rho)\right)^2\right) = O\,(1)\,\tau(\rho)^2 \\ E\left(R_{2i}^2\right) &\leq \sigma_\alpha^2\sigma_{i\varepsilon}^2\iota_T' C_T(\rho)\, C_T(\rho)'\,\iota_T = O\,(1) \\ E\left(R_{3i}^2\right) &\leq \sigma_\alpha^2\sigma_{i\varepsilon}^2\iota_T' A_T(\rho)\, A_T(\rho)'\,\iota_T\tau(\rho) = O\,(1)\,\tau(\rho) \\ E\left(R_{4i}^2\right) &= \sigma_{i\varepsilon}^4 A_T(\rho)'\, C_T(\rho)\, C_T(\rho)'\, A_T(\rho)\,\tau(\rho) = O\,(1)\,\tau(\rho)\end{aligned}$$

and

$$\begin{aligned}E\left(Q_{1i}^2\right) &= \sigma_{i\varepsilon}^4\,\mathrm{tr}\left(C_T(\rho)'\, C_T(\rho)\right) = O\,(1) \\ E\left(Q_{2i}^2\right) &= E\left(\alpha_i^4\right)(1-\rho)^2\, T^2 = O\left(N^{-2k}\right) \\ E\left(Q_{3i}^2\right) &= \sigma_\alpha^2\sigma_{i\varepsilon}^2\,(1-\rho)^2\,\iota_T' C_T(\rho)\, C_T(\rho)'\,\iota_T = O\left(N^{-2k}\right) \\ E\left(Q_{4i}^2\right) &= \sigma_\alpha^2\sigma_{i\varepsilon}^2\,(1-\rho)^2\,\iota_T' A_T(\rho)\, A_T(\rho)'\,\iota_T\tau(\rho) = O\left(N^{-2k}\right)\tau(\rho)\end{aligned}$$

To show the result in (81) we first of all show that

$$(i) \quad : \quad \frac{1}{N}\sum_{i=1}^{N} Q_{1i}^2 \xrightarrow{P} T\left(\mathbf{1}_{\{c>0\}}\sigma_\alpha^2\sigma_{2\varepsilon} + \left(\tau + \frac{T-1}{2}\right)\sigma_{4\varepsilon}\right) \quad \text{as } N \to \infty$$

$$(ii) \quad : \quad \frac{1}{N^2}\sum_{i=1}^{N} Q_{1i}^2 \xrightarrow{P} \frac{1}{2\tilde{c}} T\sigma_{4\varepsilon} \quad \text{as } N \to \infty$$

The results follow by (72) and (76) respectively and Markov's Law of Large Numbers. Next, we show that $\frac{1}{N^{2k}}\sum_{i=1}^{N} y_{i,-1}'u_iu_i'y_{i,-1} - \frac{1}{N^{2k}}\sum_{i=1}^{N} Q_{1i}^2 \xrightarrow{P} 0$ as $N \to \infty$ by showing that $E\left|\frac{1}{N^{2k}}\sum_{i=1}^{N}\left(y_{i,-1}'u_iu_i'y_{i,-1} - Q_{1i}^2\right)\right|$ $\to 0$ as $N \to \infty$. We have

$$
\begin{aligned}
& E\left|\frac{1}{N^{2k}}\sum_{i=1}^{N}\left(y_{i,-1}'u_iu_i'y_{i,-1} - Q_{1i}^2\right)\right| \\
\leq\ & \frac{1}{N^{2k}}\sum_{i=1}^{N} E\left|Q_{2i}+Q_{3i}+Q_{4i}\right|^2 + 2E\left|Q_{1i}\left(Q_{2i}+Q_{3i}+Q_{4i}\right)\right| \\
\leq\ & \frac{1}{N^{2k}}\sum_{i=1}^{N}\left(4E\left(Q_{2i}^2+Q_{3i}^2+Q_{4i}^2\right) + \sqrt{2E\left(Q_{1i}^2\right)E\left(Q_{2i}^2+Q_{3i}^2+Q_{4i}^2\right)}\right) \\
=\ & \begin{cases} O\left(N^{-\frac{1}{2}}\right) & \text{when } \tau(\rho) = \tau \text{ and } k = \frac{1}{2} \\ \frac{1}{N}O(1) & \text{when } \tau(\rho) = \frac{1}{1-\rho^2} \text{ and } k = 1 \end{cases}
\end{aligned}
$$

where the first inequality results from the triangle inequality, the second inequality results from the Cauchy-Schwarz inequality and Lemma 2 and the last line holds according to the expressions for $E\left(Q_{ki}^2\right)$ for $k = 1,2,3,4$ given above. Thus, $E\left|\frac{1}{N^{2k}}\sum_{i=1}^{N}\left(y_{i,-1}'u_iu_i'y_{i,-1} - Q_{1i}^2\right)\right| \to 0$ as $N \to \infty$. Altogether this proves the result in (81).

To show (83) and (84) we show that $E\left|\frac{1}{N^2}\sum_{i=1}^{N}\left(y_{i,-1}'y_{i,-1}\right)^2\right| \to 0$ and $E\left|\frac{1}{N^{\frac{3}{2}}}\sum_{i=1}^{N} y_{i,-1}'y_{i,-1}y_{i,-1}'u_i\right| \to$ 0 as $N \to \infty$. We have

$$
\begin{aligned}
E\left|\frac{1}{N^{4k}}\sum_{i=1}^{N}\left(y_{i,-1}'y_{i,-1}\right)^2\right| &= \frac{1}{N^{4k}}\sum_{i=1}^{N} E\left(\left(R_{1i}+R_{2i}+R_{3i}+R_{4i}\right)^2\right) \\
&\leq \frac{1}{N^{4k}}\sum_{i=1}^{N}\left(E\left(R_{1i}^2\right)^{\frac{1}{2}} + E\left(R_{2i}^2\right)^{\frac{1}{2}} + E\left(R_{3i}^2\right)^{\frac{1}{2}} + E\left(R_{4i}^2\right)^{\frac{1}{2}}\right)^2 \\
&= \begin{cases} \frac{1}{N}O(1) & \text{when } \tau(\rho) = \tau \text{ and } k = \frac{1}{2} \\ \frac{1}{N^{3k}}O\left(N^{2k}\right) & \text{when } \tau(\rho) = \frac{1}{1-\rho^2} \text{ and } k = 1 \end{cases}
\end{aligned}
$$

where the inequality results from Minkowski's inequality and the last line holds according to the expressions for $E\left(R_{ki}^2\right)$ for $k = 1,2,3,4$ given above. Thus $E\left|\frac{1}{N^{4k}}\sum_{i=1}^{N}\left(y_{i,-1}'y_{i,-1}\right)^2\right| \to 0$ as $N \to \infty$. Finally, we have

$$
\begin{aligned}
E\left|\frac{1}{N^{3k}}\sum_{i=1}^{N} y_{i,-1}'y_{i,-1}y_{i,-1}'u_i\right| &\leq \frac{1}{N^{3k}}\sum_{i=1}^{N}\sqrt{E\left(\left(y_{i,-1}'y_{i,-1}\right)^2\right)\left(E\left(y_{i,-1}'u_i\right)^2\right)} \\
&\leq \frac{1}{N^{3k}}\sum_{i=1}^{N}\left(\sum_{k=1}^{4} E\left(R_{ki}^2\right)^{\frac{1}{2}}\right)\left(\sum_{k=1}^{4} E\left(Q_{ki}^2\right)^{\frac{1}{2}}\right) \\
&= \begin{cases} \frac{1}{\sqrt{N}}O(1) & \text{when } \tau(\rho) = \tau \text{ and } k = \frac{1}{2} \\ \frac{1}{N^{2k}}O\left(N^{k}\right) & \text{when } \tau(\rho) = \frac{1}{1-\rho^2} \text{ and } k = 1 \end{cases}
\end{aligned}
$$

where the first inequality results from the triangle inequality and the Cauchy-Schwarz inequality, the second inequality results from Minkowski's inequality and the last line holds according to the expressions for $E\left(R_{ki}^2\right)$ and $E\left(Q_{ki}^2\right)$ for $k = 1,2,3,4$ given above. Thus, $E\left|\frac{1}{N^{3k}}\sum_{i=1}^{N} y_{i,-1}'y_{i,-1}y_{i,-1}'u_i\right| \to 0$ as $N \to \infty$. Altogether, we have obtained the desired limits and the result is proved. □

A.3 Proofs of the propositions in Section 3.2: Breitung-Meyer

For $-1 < \rho \leq 1$ the following expression for $y_{it} - y_{i0}$ is obtained by recursive substitution in (16)

$$y_{it} - y_{i0} = \left(\rho^t - 1\right)(y_{i0} - \alpha_i) + \rho^{t-1}\varepsilon_{i1} + ... + \varepsilon_{it} \quad \text{for } t = 1, ..., T \tag{85}$$

Inserting the expression for the initial value given in Assumption 3 yields

$$y_{it} - y_{i0} = \left(\rho^t - 1\right)\sqrt{\tau(\rho)}\varepsilon_{i0} + \rho^{t-1}\varepsilon_{i1} + ... + \varepsilon_{it} \quad \text{for } t = 1, ..., T \tag{86}$$

Using stacked notation, the regressor $\tilde{y}_{i,-1}$ and the regression error v_i can be expressed as follows

$$\begin{aligned} \tilde{y}_{i,-1} &= y_{i,-1} - \iota_T y_{i0} = C_T(\rho)\varepsilon_i + B_T(\rho)\sqrt{\tau(\rho)}\varepsilon_{i0} & (87)\\ v_i &= \varepsilon_i + (\rho - 1)\sqrt{\tau(\rho)}\iota_T\varepsilon_{i0} & (88) \end{aligned}$$

where ι_T is a $T \times 1$ vector of ones, $C_T(\rho)$ is the $T \times T$ matrix defined in (46) and $B_T(\rho)$ is the $T \times 1$ vector defined as $B_T(\rho) = \left[0, \rho - 1, ..., \rho^{T-1} - 1\right]'$. Note that $B_T(\rho) = -c/\sqrt{N}\,[0, 1, ..., T-1] + o\left(N^{-\frac{1}{2}}\right)$ when $\rho = 1 - c/\sqrt{N}$ according to Lemma 1.

Proof of Proposition 3:

Using the equation in (17) we have that

$$\sqrt{N}(\hat{\rho}_0 - \rho) = \left(\frac{1}{N}\sum_{i=1}^{N}\tilde{y}_{i,-1}'\tilde{y}_{i,-1}\right)^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\tilde{y}_{i,-1}'v_i \tag{89}$$

Proposition 3 now follows by the results in Lemma 9 below.

Lemma 9 *Under Assumption 1-4 and the local-to-unity sequence for ρ given by $\rho = 1 - c/\sqrt{N}$, the following results hold*

$$\begin{aligned} (i) + (ii) &: \frac{1}{N}\sum_{i=1}^{N}\tilde{y}_{i,-1}'\tilde{y}_{i,-1} \xrightarrow{P} \sigma_{2\varepsilon}\frac{T(T-1)}{2} \quad \textit{as } N \to \infty & \text{(a)}\\ (i) &: \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\tilde{y}_{i,-1}'v_i \xrightarrow{w} N\left(0, \sigma_{4\varepsilon}\frac{T(T-1)}{2}\right) \quad \textit{as } N \to \infty & \text{(b1)}\\ (ii) &: \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\tilde{y}_{i,-1}'v_i \xrightarrow{w} N\left(\frac{c}{2}\sigma_{2\varepsilon}\frac{T(T-1)}{2}, \sigma_{4\varepsilon}\frac{T(T-1)}{2}\right) \quad \textit{as } N \to \infty & \text{(b2)} \end{aligned}$$

Proof of Lemma 9:

(a) Using the expression for $\tilde{y}_{i,-1}$ given in equation (87) we have

$$\tilde{y}_{i,-1}'\tilde{y}_{i,-1} = R_{1i} + R_{2i} + 2R_{3i} \tag{90}$$

where

$$\begin{aligned} R_{1i} &= \varepsilon_i' C_T(\rho)' C_T(\rho)\varepsilon_i\\ R_{2i} &= \varepsilon_{i0}^2\tau(\rho) B_T(\rho)' B_T(\rho)\\ R_{3i} &= \varepsilon_i' C_T(\rho)' B_T(\rho)\sqrt{\tau(\rho)}\varepsilon_{i0} \end{aligned}$$

We prove the result by showing that

$$\frac{1}{N}\sum R_{1i} \xrightarrow{P} \sigma_{2\varepsilon}\frac{T\left(T-1\right)}{2} \quad \text{as } N \to \infty \tag{91}$$

$$\frac{1}{N}\sum R_{ki} \xrightarrow{P} 0 \quad \text{as } N \to \infty \;\; \text{for } k=2,3 \tag{92}$$

We have that

$$\frac{1}{N}\sum_{i=1}^{N} E\left(R_{1i}\right) = \frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{2}\operatorname{tr}\left(C_T\left(\rho\right)' C_T\left(\rho\right)\right) \to \sigma_{2\varepsilon}\frac{T\left(T-1\right)}{2} \quad \text{as } N \to \infty$$

since $\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{2} \to \sigma_{2\varepsilon}$ as $N \to \infty$ by Assumption 4 and $\operatorname{tr}\left(C_T\left(\rho\right)' C_T\left(\rho\right)\right) = \operatorname{tr}\left(C_T\left(1\right)' C_T\left(1\right)\right) + O\left(N^{-\frac{1}{2}}\right) = \frac{T(T-1)}{2} + O\left(N^{-\frac{1}{2}}\right)$ by Lemma 1. The result in (91) then follows by Markov's Law of Large Numbers which can be applied since $E\left|R_{1i}\right|^{1+\delta_1} \le E\left|R_{1i}\right|^{2+\delta_1} + 1 < K_1 + 1$ for all $i = 1, ..., N$ according to Lemma 2 and (47) in Lemma 4.

Next, we show (92) by showing that $E\left|\frac{1}{N}\sum_{i=1}^{N} R_{ki}\right| \to 0$ as $N \to \infty$ for $k = 2, 3$. We have that

$$\begin{aligned} E\left|\frac{1}{N}\sum_{i=1}^{N} R_{2i}\right| &\le \frac{1}{N}\sum_{i=1}^{N} E\left(\varepsilon_{i0}^{2}\right)\tau\left(\rho\right) B_T\left(\rho\right)' B_T\left(\rho\right) \\ &= \frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{2}\tau\left(\rho\right) B_T\left(\rho\right)' B_T\left(\rho\right) = O\left(N^{-\frac{1}{2}}\right) \end{aligned} \tag{93}$$

where the first inequality results from the triangle inequality and the last equality sign holds as $\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{2} = O\left(1\right)$ by Assumption 4 and $\tau\left(\rho\right) = O\left(N^{\frac{1}{2}}\right)$ and $B_T\left(\rho\right) = O\left(N^{-\frac{1}{2}}\right)$ by Lemma 1. Thus, $E\left|\frac{1}{N}\sum_{i=1}^{N} R_{2i}\right| \to 0$ as $N \to \infty$. Finally, we have that

$$\begin{aligned} E\left|\frac{1}{N}\sum_{i=1}^{N} R_{3i}\right| &\le \frac{1}{N}\sum_{i=1}^{N} E\left|\varepsilon_{i0}\varepsilon_i' C_T\left(\rho\right)' B_T\left(\rho\right)\sqrt{\tau\left(\rho\right)}\right| \\ &\le \frac{1}{N}\sum_{i=1}^{N}\sqrt{E\left(\varepsilon_{i0}^{2}\right) E\left(\varepsilon_i' C_T\left(\rho\right)' B_T\left(\rho\right) B_T\left(\rho\right)' C_T\left(\rho\right)\varepsilon_i\right)\tau\left(\rho\right)} \\ &= \frac{1}{N}\sum_{i=1}^{N}\sqrt{E\left(\varepsilon_{i0}^{2}\right) E\left(\varepsilon_{it}^{2}\right)\operatorname{tr}\left(C_T\left(\rho\right)' B_T\left(\rho\right) B_T\left(\rho\right)' C_T\left(\rho\right)\right)\tau\left(\rho\right)} \\ &= \frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{4}\sqrt{\tau\left(\rho\right) B_T\left(\rho\right)' C_T\left(\rho\right) C_T\left(\rho\right)' B_T\left(\rho\right)} = O\left(N^{-\frac{1}{4}}\right) \end{aligned} \tag{94}$$

where the first inequality results from the triangle inequality, the second inequality results from the Cauchy-Schwarz inequality and the last line holds by using the same arguments as above. Thus, $E\left|\frac{1}{N}\sum_{i=1}^{N} R_{3i}\right| \to 0$ as $N \to \infty$. Altogether, the desired results hold and part (a) is proved.

(b1) and (b2) Using the expressions for $\tilde{y}_{i,-1}$ and v_i given in equations (87)and (88) we have

$$\tilde{y}_{i,-1}' v_i = Q_{1i} + Q_{2i} + Q_{3i} + Q_{4i} \tag{95}$$

where

$$\begin{aligned} Q_{1i} &= \varepsilon_i' C_T(\rho)' \varepsilon_i \\ Q_{2i} &= \varepsilon_i' C_T(\rho)' \iota_T (\rho - 1) \sqrt{\tau(\rho)} \varepsilon_{i0} \\ Q_{3i} &= \varepsilon_i' B_T(\rho) \sqrt{\tau(\rho)} \varepsilon_{i0} \\ Q_{4i} &= \varepsilon_{i0}^2 \tau(\rho) (\rho - 1) B_T(\rho)' \iota_T \end{aligned}$$

We show that

$$(i) \quad : \quad \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (Q_{1i} + E(Q_{4i})) \xrightarrow{w} N\left(0, \sigma_{4\varepsilon} \frac{T(T-1)}{2}\right) \quad \text{as } N \to \infty \tag{96}$$

$$(ii) \quad : \quad \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (Q_{1i} + E(Q_{4i})) \xrightarrow{w} N\left(\frac{c}{2} \sigma_{2\varepsilon} \frac{T(T-1)}{2}, \sigma_{4\varepsilon} \frac{T(T-1)}{2}\right) \quad \text{as } N \to \infty \tag{97}$$

and

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} Q_{ki} \xrightarrow{P} 0 \quad \text{as } N \to \infty, \quad \text{for } k = 2, 3 \tag{98}$$

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} (Q_{4i} - E(Q_{4i})) \xrightarrow{P} 0 \quad \text{as } N \to \infty \tag{99}$$

First to show (96) and (97) we note that

$$\begin{aligned} E(Q_{1i}) &= E\left(\varepsilon_i' C_T(\rho)' \varepsilon_i\right) = \sigma_{i\varepsilon}^2 \operatorname{tr}(C_T(\rho)) = 0 \\ \operatorname{Var}(Q_{1i}) &= E\left(Q_{1i}^2\right) = \sigma_{i\varepsilon}^4 \operatorname{tr}\left(C_T(\rho)' C_T(\rho)\right) \end{aligned}$$

where we have used Lemma 6. Using this we find

$$\frac{1}{N} \sum_{i=1}^{N} \operatorname{Var}(Q_{1i}) = \frac{1}{N} \sum_{i=1}^{N} E\left(Q_{1i}^2\right) \to \sigma_{4\varepsilon} \frac{T(T-1)}{2} \quad \text{as } N \to \infty \tag{100}$$

Then the Liapounov Central Limit Theorem, which can be applied since $E|Q_{1i}|^{2+\delta_1} < K_4$ for all $i = 1, ..., N$ according to (50) in Lemma 4, gives

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} Q_{1i} \xrightarrow{w} N\left(0, \sigma_{4\varepsilon} \frac{T(T-1)}{2}\right) \quad \text{as } N \to \infty \tag{101}$$

Also we have that

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} E(Q_{4i}) = -c\left(\frac{1}{N} \sum_{i=1}^{N} \sigma_{i\varepsilon}^2\right) \tau(\rho) \left((1 + 2 + ... + (T-1)) \frac{-c}{\sqrt{N}} + o\left(N^{-\frac{1}{2}}\right)\right)$$

which implies

$$(i) \quad : \quad \frac{1}{\sqrt{N}} \sum_{i=1}^{N} E(Q_{4i}) \to 0 \quad \text{as } N \to \infty \tag{102}$$

$$(ii) \quad : \quad \frac{1}{\sqrt{N}} \sum_{i=1}^{N} E(Q_{4i}) \to \frac{c}{2} \sigma_{2\varepsilon} \frac{T(T-1)}{2} \quad \text{as } N \to \infty \tag{103}$$

Using this in combination with (101) yields the results in (96) and (97).

Next, we show (98) by showing that $E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Q_{ki}\right|^{2}\rightarrow 0$ as $N\rightarrow\infty$ for $k=2,3$. We have that

$$\begin{aligned} E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Q_{2i}\right|^{2} &= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}E\left(Q_{2i}Q_{2j}\right)=\frac{1}{N}\sum_{i=1}^{N}E\left(Q_{2i}^{2}\right) \\ &= \frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{4}\tau\left(\rho\right)\left(\rho-1\right)^{2}\iota_{T}'C_{T}\left(\rho\right)C_{T}\left(\rho\right)'\iota_{T}=O\left(N^{-\frac{1}{2}}\right) \end{aligned} \tag{104}$$

where the first line holds as the sequence Q_{2i} is independent across i with mean zero and the last line holds as $\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{4}=O\left(1\right)$, $\tau\left(\rho\right)=O\left(N^{\frac{1}{2}}\right)$, $\left(\rho-1\right)^{2}=O\left(N\right)$ and $\iota_{T}'C_{T}\left(\rho\right)C_{T}\left(\rho\right)'\iota_{T}=O\left(1\right)$. Thus $E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Q_{2i}\right|^{2}\rightarrow 0$ as $N\rightarrow\infty$. Using similar arguments we have

$$E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Q_{3i}\right|^{2}=\frac{1}{N}\sum_{i=1}^{N}E\left(Q_{3i}^{2}\right)=\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{4}\tau\left(\rho\right)B_{T}\left(\rho\right)'B_{T}\left(\rho\right)=O\left(N^{-\frac{1}{2}}\right) \tag{105}$$

such that $E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Q_{3i}\right|^{2}\rightarrow 0$ as $N\rightarrow\infty$. Finally we show (99) by showing that $E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(Q_{4i}-E\left(Q_{4i}\right)\right)\right|^{2}\rightarrow 0$ as $N\rightarrow\infty$. We have

$$\begin{aligned} E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(Q_{4i}-E\left(Q_{4i}\right)\right)\right|^{2} &= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\text{Cov}\left(Q_{4i},Q_{4j}\right)\leq\frac{1}{N}\sum_{i=1}^{N}E\left(Q_{4i}^{2}\right) \\ &= \frac{1}{N}\sum_{i=1}^{N}E\left(\varepsilon_{i0}^{4}\right)\tau\left(\rho\right)^{2}\left(\rho-1\right)^{2}B_{T}\left(\rho\right)'\iota_{T}=O\left(N^{-\frac{1}{2}}\right) \end{aligned} \tag{106}$$

The first line holds as the sequence $Q_{4i}-E\left(Q_{4i}\right)$ is independent across i with mean zero. The rest follows by using the same arguments as above. Altogether, the desired results hold and part (b1) and (b2) is proved. $\square$

Proof of Proposition 4:

The proposition follows by the results already obtained in the previous and Lemma 10 given below.

Lemma 10 *Under Assumption 1-4 and the local-to-unity sequence for ρ given by $\rho=1-c/\sqrt{N}$ for $c\geq 0$ the following result holds*

$$\sum_{i=1}^{N}\tilde{y}_{i,-1}'\hat{v}_{i}\hat{v}_{i}'\tilde{y}_{i,-1}\xrightarrow{P}\sigma_{4\varepsilon}\frac{T\left(T-1\right)}{2}\quad \textit{as } N\rightarrow\infty \tag{a}$$

Combining the result in Lemma 10 with (a) in Lemma 9 we have that

$$\begin{aligned} \hat{V}_{0} &= \left(\frac{1}{N}\sum_{i=1}^{N}\tilde{y}_{i,-1}'\tilde{y}_{i,-1}\right)^{-1}\frac{1}{N}\sum_{i=1}^{N}\tilde{y}_{i,-1}'\hat{v}_{i}\hat{v}_{i}'\tilde{y}_{i,-1}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{y}_{i,-1}'\tilde{y}_{i,-1}\right)^{-1} \\ &\xrightarrow{P}\frac{\sigma_{4\varepsilon}}{\sigma_{2\varepsilon}^{2}}\frac{T\left(T-1\right)}{2}\quad \textit{as } N\rightarrow\infty \end{aligned} \tag{107}$$

Using the expressions for t_0 and $\bar{t}_0$ given in (21) and (22) respectively we have that

$$\begin{aligned} t_0 &= \hat{V}_0^{-\frac{1}{2}}\sqrt{N}\left(\hat{\rho}_0-1\right)=\hat{V}_0^{-\frac{1}{2}}\sqrt{N}\left(\hat{\rho}_0-\rho\right)-c\hat{V}_0^{-\frac{1}{2}} &(108)\\ \bar{t}_0 &= \sqrt{\frac{T\left(T-1\right)}{2}}\sqrt{N}\left(\hat{\rho}_0-1\right)=\sqrt{\frac{T\left(T-1\right)}{2}}\sqrt{N}\left(\hat{\rho}_0-\rho\right)-c\sqrt{\frac{T\left(T-1\right)}{2}} &(109)\end{aligned}$$

Using this the results in Proposition 3 yields the results in Proposition 4.

Proof of Lemma 10:

Inserting the expression for $\hat{v}_i$ given by $\hat{v}_i=v_i+\left(\rho-\hat{\rho}_0\right)\tilde{y}_{i,-1}$ yields

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}\tilde{y}_{i,-1}'\hat{v}_i\hat{v}_i'\tilde{y}_{i,-1} &= \frac{1}{N}\sum_{i=1}^{N}\tilde{y}_{i,-1}'v_iv_i'\tilde{y}_{i,-1}+\left(\rho-\hat{\rho}_0\right)^2\frac{1}{N}\sum_{i=1}^{N}\tilde{y}_{i,-1}'\tilde{y}_{i,-1}\tilde{y}_{i,-1}'\tilde{y}_{i,-1}\\ &\quad +2\left(\rho-\hat{\rho}_0\right)\frac{1}{N}\sum_{i=1}^{N}\tilde{y}_{i,-1}'\tilde{y}_{i,-1}\tilde{y}_{i,-1}'v_i\end{aligned}$$

As $\sqrt{N}\left(\hat{\rho}_0-\rho\right)=O\left(1\right)$ by Proposition 3, we prove the result by showing that

$$\frac{1}{N}\sum_{i=1}^{N}\tilde{y}_{i,-1}'v_iv_i'\tilde{y}_{i,-1}\xrightarrow{P}\sigma_{4\varepsilon}\frac{T\left(T-1\right)}{2}\quad\text{as } N\to\infty \tag{110}$$

$$\frac{1}{N^2}\sum_{i=1i}^{N}\left(\tilde{y}_{i,-1}'\tilde{y}_{i,-1}\right)^2\xrightarrow{P}0\quad\text{as } N\to\infty \tag{111}$$

$$\frac{1}{N^{\frac{3}{2}}}\sum_{i=1}^{N}\tilde{y}_{i,-1}'\tilde{y}_{i,-1}\tilde{y}_{i,-1}'v_i\xrightarrow{P}0\quad\text{as } N\to\infty \tag{112}$$

According to (90) we have $\tilde{y}_{i,-1}'\tilde{y}_{i,-1}=\sum_{k=1}^{3}R_{ki}$ and according to (95) we have $\tilde{y}_{i,-1}'v_i=\sum_{i=1}^{4}Q_{ki}$. Using these expressions we have

$$\begin{aligned} E\left(R_{1i}^2\right) &= E\left(\left(\varepsilon_i'C_T\left(\rho\right)'C_T\left(\rho\right)\varepsilon_i\right)^2\right)=O\left(1\right)\\ E\left(R_{2i}^2\right) &= E\left(\varepsilon_{i0}^4\right)\tau\left(\rho\right)^2\left(B_T\left(\rho\right)'B_T\left(\rho\right)\right)^2=O\left(N^{-1}\right)\\ E\left(R_{3i}^2\right) &= \sigma_{i\varepsilon}^4B_T\left(\rho\right)'C_T\left(\rho\right)C_T\left(\rho\right)'B_T\left(\rho\right)\tau\left(\rho\right)=O\left(N^{-\frac{1}{2}}\right)\end{aligned}$$

and

$$\begin{aligned} E\left(Q_{1i}^2\right) &= E\left(\left(\varepsilon_i'C_T\left(\rho\right)'\varepsilon_i\right)^2\right)=\sigma_{i\varepsilon}^4\operatorname{tr}\left(C_T\left(\rho\right)'C_T\left(\rho\right)\right)=O\left(1\right)\\ E\left(Q_{2i}^2\right) &= \sigma_{i\varepsilon}^4\left(\rho-1\right)^2\tau\left(\rho\right)\iota_T'C_T\left(\rho\right)C_T\left(\rho\right)'\iota_T=O\left(N^{-\frac{1}{2}}\right)\\ E\left(Q_{3i}^2\right) &= \sigma_{i\varepsilon}^4\tau\left(\rho\right)B_T\left(\rho\right)'B_T\left(\rho\right)=O\left(N^{-\frac{1}{2}}\right)\\ E\left(Q_{4i}^2\right) &= E\left(\varepsilon_{i0}^4\right)\tau\left(\rho\right)^2\left(\rho-1\right)^2B_T\left(\rho\right)'\iota_T\iota_T'B_T\left(\rho\right)=O\left(N^{-1}\right)\end{aligned}$$

To show the result in (110) we first of all show that

$$\frac{1}{N}\sum_{i=1}^{N}Q_{1i}^2=\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i'C_T\left(\rho\right)'\varepsilon_i\varepsilon_i'C_T\left(\rho\right)\varepsilon_i\xrightarrow{P}\sigma_{4\varepsilon}\frac{T\left(T-1\right)}{2}\quad\text{as } N\to\infty$$

This follows by (100) and Markov's Law of Large Numbers which can be applied since $E\left|Q_{1i}^2\right|^{1+\delta_1/2}=E\left|Q_{1i}\right|^{2+\delta_1}<K_4$ for all $i=1,...,N$ according to (50) in Lemma 4. Next, we show that

$\frac{1}{N}\sum_{i=1}^{N}\tilde{y}_{i,-1}'v_iv_i'\tilde{y}_{i,-1}-\frac{1}{N}\sum_{i=1}^{N}Q_{1i}^2\xrightarrow{P}0$ as $N\to\infty$ by showing that $E\left|\frac{1}{N}\sum_{i=1}^{N}\left(\tilde{y}_{i,-1}'v_iv_i'\tilde{y}_{i,-1}-Q_{1i}^2\right)\right|$ $\to 0$ as $N\to\infty$. We have

$$\begin{aligned}
E\left|\frac{1}{N}\sum_{i=1}^{N}\left(\tilde{y}_{i,-1}'v_iv_i'\tilde{y}_{i,-1}-Q_{1i}^2\right)\right| &\leq \frac{1}{N}\sum_{i=1}^{N}E\left|Q_{2i}+Q_{3i}+Q_{4i}\right|^2+2E\left|Q_{1i}\left(Q_{2i}+Q_{3i}+Q_{4i}\right)\right| \\
&\leq \frac{1}{N}\sum_{i=1}^{N}\left(4E\left(Q_{2i}^2+Q_{3i}^2+Q_{4i}^2\right)+\sqrt{2E\left(Q_{1i}^2\right)E\left(Q_{2i}^2+Q_{3i}^2+Q_{4i}^2\right)}\right) \\
&= O\left(N^{-\frac{1}{2}}\right)
\end{aligned}$$

where the first inequality results from the triangle inequality, the second inequality results from the Cauchy-Schwarz inequality and Lemma 2 and the last line holds according to the expressions for $E\left(Q_{ki}^2\right)$ for $k=1,2,3,4$ given above. Thus, $E\left|\frac{1}{N}\sum_{i=1}^{N}\left(\tilde{y}_{i,-1}'v_iv_i'\tilde{y}_{i,-1}-Q_{1i}^2\right)\right|\to 0$ as $N\to\infty$. Altogether this proves the result in (110).

To show (111) and (112) we show that $E\left|\frac{1}{N^2}\sum_{i=1}^{N}\left(\tilde{y}_{i,-1}'\tilde{y}_{i,-1}\right)^2\right|\to 0$ and $E\left|\frac{1}{N^{\frac{3}{2}}}\sum_{i=1}^{N}\tilde{y}_{i,-1}'\tilde{y}_{i,-1}\tilde{y}_{i,-1}'v_i\right|$ $\to 0$ as $N\to\infty$. We have

$$\begin{aligned}
E\left|\frac{1}{N^2}\sum_{i=1}^{N}\left(\tilde{y}_{i,-1}'\tilde{y}_{i,-1}\right)^2\right| &= \frac{1}{N^2}\sum_{i=1}^{N}E\left(\left(R_{1i}+R_{2i}+R_{3i}\right)^2\right) \\
&\leq 4\frac{1}{N^2}\sum_{i=1}^{N}\left(E\left(R_{1i}^2\right)+E\left(R_{2i}^2\right)+E\left(R_{3i}^2\right)\right) \\
&= 2\frac{1}{N}O(1)
\end{aligned}$$

where the inequality follows by Lemma 2 and the last line holds according to the expressions for $E\left(R_{ki}^2\right)$ for $k=1,2,3$ given above. Thus $E\left|\frac{1}{N^2}\sum_{i=1}^{N}\left(\tilde{y}_{i,-1}'\tilde{y}_{i,-1}\right)^2\right|\to 0$ as $N\to\infty$. Finally, we have

$$\begin{aligned}
E\left|\frac{1}{N^{\frac{3}{2}}}\sum_{i=1}^{N}\tilde{y}_{i,-1}'\tilde{y}_{i,-1}\tilde{y}_{i,-1}'v_i\right| &\leq \frac{1}{N^{\frac{3}{2}}}\sum_{i=1}^{N}\sqrt{E\left(\left(\tilde{y}_{i,-1}'\tilde{y}_{i,-1}\right)^2\right)\left(E\left(\tilde{y}_{i,-1}'v_i\right)^2\right)} \\
&\leq \frac{1}{N^{\frac{3}{2}}}\sum_{i=1}^{N}\left(\sum_{k=1}^{3}E\left(R_{ki}^2\right)^{\frac{1}{2}}\right)\left(\sum_{k=1}^{4}E\left(Q_{ki}^2\right)^{\frac{1}{2}}\right) \\
&= \frac{1}{\sqrt{N}}O(1)
\end{aligned}$$

where the first inequality results from the triangle inequality and the Cauchy-Schwarz inequality, the second inequality results from Minkowski's inequality and the last line holds according to the expressions for $E\left(R_{ki}^2\right)$ for $k=1,2,3$ and $E\left(Q_{ki}^2\right)$ for $k=1,2,3,4$ given above. Thus, $E\left|\frac{1}{N^{\frac{3}{2}}}\sum_{i=1}^{N}\tilde{y}_{i,-1}'\tilde{y}_{i,-1}\tilde{y}_{i,-1}'v_i\right|\to$ 0 as $N\to\infty$. Altogether, we have obtained the desired limits and the result is proved. $\square$

A.4 Proofs of the propositions in Section 3.3: Harris-Tzavalis

Using the expressions for $y_{i,-1}$ and u_i given in (58) and (59) and that $Q_T\iota_T=0$ we have

$$Q_Ty_{i,-1} = Q_TC_T(\rho)\varepsilon_i+Q_TA_T(\rho)\sqrt{\tau(\rho)}\varepsilon_{i0} \tag{113}$$

$$Q_Tu_i = Q_T\varepsilon_i \tag{114}$$

where Q_T is the symmetric idempotent $T \times T$ matrix defined as $Q_T = I_T - \frac{1}{T}\iota_T\iota_T'$ and $C_T(\rho)$ is the $T \times T$ matrix and $A_T(\rho)$ is the $T \times 1$ vector both defined in (46).

Proof of Proposition 5:

Using the expression for $\hat{\rho}_{WG}$ in equation (28) we have

$$\begin{aligned} &\sqrt{N}\left(\hat{\rho}_{WG} - \rho + \frac{3}{T+1}\right) \\ = &\left(\frac{1}{N}\sum_{i=1}^{N} y_{i,-1}' Q_T y_{i,-1}\right)^{-1} \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(y_{i,-1}' Q_T \varepsilon_i + \frac{3}{T+1} y_{i,-1}' Q_T y_{i,-1}\right) \end{aligned} \tag{115}$$

Proposition 5 now follows by the results in Lemma 11 below.

Lemma 11 *Under Assumption 1, 2, 3, 4 and the local-to-unity sequence for ρ given by $\rho = 1 - c/\sqrt{N}$ the following results hold*

$$\frac{1}{N}\sum_{i=1}^{N} y_{i,-1}' Q_T y_{i,-1} \xrightarrow{P} \sigma_{2\varepsilon}\frac{(T-1)(T+1)}{6} \quad \text{as } N \to \infty \tag{a}$$

$$(i) : \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(y_{i,-1}' Q_T \varepsilon_i + \frac{3}{T+1} y_{i,-1}' Q_T y_{i,-1}\right) \xrightarrow{w} N\left(-c\sigma_{2\varepsilon} b_1, g_1 m_4 + g_2 \sigma_{4\varepsilon}\right) \quad \text{as } N \to \infty \tag{b1}$$

$$(ii) : \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(y_{i,-1}' Q_T \varepsilon_i + \frac{3}{T+1} y_{i,-1}' Q_T y_{i,-1}\right) \xrightarrow{w} N\left(c\sigma_{2\varepsilon} b_2, g_1 m_4 + g_2 \sigma_{4\varepsilon}\right) \quad \text{as } N \to \infty \tag{b2}$$

where

$$b_1 = \frac{(T-1)(T-2)}{12} \qquad b_2 = \frac{(T-1)(T+4)}{24} \tag{116}$$

and

$$g_1 = \frac{(T-1)(T-2)(2T-1)}{15T(T+1)} \qquad g_2 = \frac{(T-1)\left(17T^3 - 44T^2 + 77T - 24\right)}{60T(T+1)} \tag{117}$$

Proof of Lemma 11:

(a) Using the expression for $Q_T y_{i,-1}$ given in equation (113) above we have

$$y_{i,-1}' Q_T y_{i,-1} = R_{1i} + R_{2i} + 2R_{3i} \tag{118}$$

where

$$\begin{aligned} R_{1i} &= \varepsilon_i' C_T(\rho)' Q_T C_T(\rho)\varepsilon_i \\ R_{2i} &= \varepsilon_{i0}^2 \tau(\rho) A_T(\rho)' Q_T A_T(\rho) \\ R_{3i} &= \varepsilon_i' C_T(\rho)' Q_T A_T(\rho)\sqrt{\tau(\rho)}\varepsilon_{i0} \end{aligned}$$

We prove the result by showing that

$$\frac{1}{N}\sum_{i=1}^{N} R_{1i} \xrightarrow{P} \sigma_{2\varepsilon}\frac{(T-1)(T+1)}{6} \quad \text{as } N \to \infty \tag{119}$$

$$\frac{1}{N}\sum_{i=1}^{N} R_{ki} \xrightarrow{P} 0 \quad \text{as } N \to \infty \quad \text{for } k = 2, 3 \tag{120}$$

The result in (119) follows by Markov's Large of Large Numbers which can be applied according to (47) and (49) in Lemma 4. This gives

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N} R_{1i} &= \frac{1}{N}\sum_{i=1}^{N} \varepsilon_i' C_T(\rho)' C_T(\rho)\varepsilon_i - \frac{1}{T}\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i' C_T(\rho)' \iota_T \iota_T' C_T(\rho)\varepsilon_i \\ &\xrightarrow{P} \sigma_{2\varepsilon}\left(\frac{T(T-1)}{2} - \frac{1}{T}\frac{T(T-1)(2T-1)}{6}\right) = \sigma_{2\varepsilon}\frac{(T-1)(T+1)}{6} \quad \text{as } N\to\infty\end{aligned}$$

Next, we show (120) by showing that $E\left|\frac{1}{N}\sum_{i=1}^{N} R_{ki}\right| \to 0$ as $N\to\infty$ for $k=2,3$. We have that

$$\begin{aligned}E\left|\frac{1}{N}\sum_{i=1}^{N} R_{2i}\right| &\le \frac{1}{N}\sum_{i=1}^{N} E\left(\varepsilon_{i0}^2\right)\tau(\rho) A_T(\rho)' Q_T A_T(\rho) \\ &= \frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^2 \tau(\rho) A_T(\rho)' Q_T A_T(\rho) = O\left(N^{-\frac{1}{2}}\right)\end{aligned}$$

where the inequality follows by the triangle inequality and the last equality sign holds as $\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^2 = O(1)$, $\tau(\rho) = O\left(N^{\frac{1}{2}}\right)$ and $A_T(\rho)' Q_T A_T(\rho) = O\left(N^{-1}\right)$. The last valuation follows as $A_T(\rho) = \iota_T + O\left(N^{-\frac{1}{2}}\right)$ and $Q_T\iota_T = 0$. Thus, $E\left|\frac{1}{N}\sum_{i=1}^{N} R_{2i}\right| \to 0$ as $N\to\infty$. We also have that

$$\begin{aligned}E\left|\frac{1}{N}\sum_{i=1}^{N} R_{3i}\right| &\le \frac{1}{N}\sum_{i=1}^{N}\sqrt{E\left(\varepsilon_i' C_T(\rho)' Q_T A_T(\rho) A_T(\rho)' Q_T C_T(\rho)\right)\tau(\rho) E\left(\varepsilon_{i0}^2\right)} \\ &= \frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^2\sqrt{A_T(\rho)' Q_T C_T(\rho) C_T(\rho)' Q_T A_T(\rho)\tau(\rho)} = O\left(N^{-\frac{1}{4}}\right)\end{aligned}$$

where the inequality follows from the triangle inequality and the Cauchy-Schwarz inequality and the last equality sign holds as $\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^2 = O(1)$, $\tau(\rho) = O\left(N^{\frac{1}{2}}\right)$, $C_T(\rho) = O(1)$ and $A_T(\rho)' Q_T = O\left(N^{-\frac{1}{2}}\right)$. Thus, $E\left|\frac{1}{N}\sum_{i=1}^{N} R_{3i}\right| \to 0$ as $N\to\infty$. Altogether, we have obtained the desired limits and the result in (a) is proved.

(b1) and (b2) Using the expression for $Q_T y_{i,-1}$ given in (113) we have

$$y_{i,-1}' Q_T \varepsilon_i = Q_{1i} + Q_{2i} \tag{121}$$

where

$$\begin{aligned}Q_{1i} &= \varepsilon_i' C_T(\rho)' Q_T \varepsilon_i \\ Q_{2i} &= \varepsilon_i' Q_T A_T(\rho)\sqrt{\tau(\rho)}\varepsilon_{i0}\end{aligned}$$

We show the results in (b1) and (b2) by showing that

$$(i) \quad : \quad \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(Q_{1i} + \frac{3}{T+1}\left(R_{1i} + E\left(R_{2i}\right)\right)\right) \xrightarrow{w} N\left(-c\sigma_{2\varepsilon} b_1, g_1 m_4 + g_2\sigma_{4\varepsilon}\right) \quad \text{as } N\to\infty \tag{122}$$

$$(ii) \quad : \quad \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(Q_{1i} + \frac{3}{T+1}\left(R_{1i} + E\left(R_{2i}\right)\right)\right) \xrightarrow{w} N\left(c\sigma_{2\varepsilon} b_2, g_1 m_4 + g_2\sigma_{4\varepsilon}\right) \quad \text{as } N\to\infty \tag{123}$$

and

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Q_{2i}\xrightarrow{P}0 \quad \text{as } N\to\infty \tag{124}$$

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(R_{2i}-E\left(R_{2i}\right)\right)\xrightarrow{P}0 \quad \text{as } N\to\infty \tag{125}$$

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}R_{3i}\xrightarrow{P}0 \quad \text{as } N\to\infty \tag{126}$$

To show (122) and (123) we use that according to Lemma 1

$$\begin{aligned} A_T(\rho) &= \iota_T-\tilde{A}_T\frac{c}{\sqrt{N}}+o\left(N^{-\frac{1}{2}}\right) && (127)\\ C_T(\rho) &= C_T(1)-\tilde{C}_T\frac{c}{\sqrt{N}}+o\left(N^{-\frac{1}{2}}\right) && (128)\end{aligned}$$

where the $T\times 1$ vector $\tilde{A}_T$ and the $T\times T$ matrix $\tilde{C}_T$ are defined as

$$\tilde{A}_T=\begin{bmatrix}0\\1\\\vdots\\T-2\\T-1\end{bmatrix}\qquad \tilde{C}_T=\begin{bmatrix}0&0&\cdots&0&0\\0&0&\cdots&\vdots&\vdots\\1&\ddots&\ddots&\vdots&\vdots\\\vdots&\ddots&\ddots&0&\vdots\\T-2&\cdots&1&0&0\end{bmatrix} \tag{129}$$

Using this we have

$$\begin{aligned}
&E\left(Q_{1i}+\frac{3}{T+1}R_{1i}\right)\\
&= \sigma_{i\varepsilon}^2\left(\operatorname{tr}\left(C_T(\rho)\right)-\frac{1}{T}\iota_T'C_T(\rho)'\iota_T+\frac{3}{T+1}\left(\operatorname{tr}\left(C_T(\rho)'C_T(\rho)\right)-\frac{1}{T}\iota_T'C_T(\rho)'C_T(\rho)\iota_T\right)\right)\\
&= \sigma_{i\varepsilon}^2\left(\frac{3}{T+1}\left(\operatorname{tr}\left(C_T(1)'C_T(1)\right)-\frac{1}{T}\iota_T'C_T(1)'C_T(1)\iota_T\right)-\frac{1}{T}\iota_T'C_T(1)'\iota_T\right)\\
&\quad+\sigma_{i\varepsilon}^2\left(\frac{6}{T+1}\left(\frac{1}{T}\iota_T'C_T(1)'\tilde{C}_T\iota_T-\operatorname{tr}\left(C_T(1)'\tilde{C}_T\right)\right)+\frac{1}{T}\iota_T'\tilde{C}_T{}'\iota_T\right)\frac{c}{\sqrt{N}}+o\left(N^{-\frac{1}{2}}\right)\\
&= \sigma_{i\varepsilon}^2\left(\frac{3}{T+1}\frac{(T-1)(T+1)}{6}-\frac{1}{T}\frac{T(T-1)}{2}\right)+\sigma_{i\varepsilon}^2\left(-\frac{(T-1)(T-2)}{12}\right)\frac{c}{\sqrt{N}}+o\left(N^{-\frac{1}{2}}\right)\\
&= \sigma_{i\varepsilon}^2\left(-\frac{(T-1)(T-2)}{12}\right)\frac{c}{\sqrt{N}}+o\left(N^{-\frac{1}{2}}\right)
\end{aligned}$$

where we have used the following results

$$\begin{aligned}
\operatorname{tr}\left(C_T(1)'\tilde{C}_T\right) &= \frac{T(T-1)(T-2)}{6}\\
\iota_T'C_T(1)'\tilde{C}_T\iota_T &= \frac{T(T-1)(T-2)(3T-1)}{24}\\
\iota_T'\tilde{C}_T{}'\iota_T &= \frac{T(T-1)(T-2)}{6}
\end{aligned}$$

This implies that

$$(i)+(ii):\frac{1}{\sqrt{N}}\sum_{i=1}^{N}E\left(Q_{1i}+\frac{3}{T+1}R_{1i}\right)\to -c\sigma_{2\varepsilon}\frac{(T-1)(T-2)}{12} \quad \text{as } N\to\infty \tag{130}$$

Also using the expression for $A_T(\rho)$ in (127) and that $\iota_T' Q_T$ we have

$$E(R_{2i}) = \sigma_{i\varepsilon}^2 \tau(\rho) A_T(\rho)' Q_T A_T(\rho) = \sigma_{i\varepsilon}^2 \tau(\rho) \left(\tilde{A}_T' Q_T \tilde{A}_T \frac{c^2}{N} + o\left(N^{-1}\right) \right)$$

This implies that

$$(i) : \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{3}{T+1} E(R_{2i}) = \frac{3}{T+1} \frac{1}{N} \sum_{i=1}^{N} \sigma_{i\varepsilon}^2 \tau(\rho) \sqrt{N} A_T(\rho)' Q_T A_T(\rho) \to 0 \quad \text{as } N \to \infty \tag{131}$$

as $\frac{1}{N}\sum_{i=1}^{N} \sigma_{i\varepsilon}^2 = O(1)$, $\tau(\rho) = O(1)$ and $A_T(\rho)' Q_T A_T(\rho) = O\left(N^{-1}\right)$ such that$\sqrt{N} A_T(\rho)' Q_T A_T(\rho) = O\left(N^{-\frac{1}{2}}\right)$. Also we have that

$$\begin{aligned} (ii) : \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{3}{T+1} E(R_{2i}) &= \frac{3}{T+1} \frac{1}{N} \sum_{i=1}^{N} \sigma_{i\varepsilon}^2 \frac{\sqrt{N}}{2c} \sqrt{N} \tilde{A}_T' Q_T \tilde{A}_T \frac{c^2}{N} + o(1) \\ &\to \sigma_{2\varepsilon} \frac{c}{2} \frac{T(T-1)}{4} \quad \text{as } N \to \infty \end{aligned} \tag{132}$$

as $\tau(\rho) = \frac{\sqrt{N}}{2c}$ according to Lemma 1 and the following result

$$\tilde{A}_T' Q_T \tilde{A}_T = \tilde{A}_T' \tilde{A}_T - \frac{1}{T}\left(\tilde{A}_T' \iota_T\right)^2 = \frac{T(T-1)(T+1)}{12}$$

We also have that

$$\begin{aligned} & \text{Var}\left(Q_{1i} + \frac{3}{T+1} R_{1i}\right) \\ = \; & E\left(Q_{1i}^2\right) + \left(\frac{3}{T+1}\right)^2 E\left(R_{1i}^2\right) + \frac{6}{T+1} E\left(Q_{1i} R_{1i}\right) + O\left(N^{-1}\right) \\ = \; & E\left(\left(\varepsilon_i' C_T(1)' Q_T \varepsilon_i\right)^2\right) + \left(\frac{3}{T+1}\right)^2 E\left(\left(\varepsilon_i' C_T(1)' Q_T C_T(1) \varepsilon_i\right)^2\right) \\ & + \frac{6}{T+1} E\left(\varepsilon_i' C_T(1)' Q_T \varepsilon_i \varepsilon_i' C_T(1)' Q_T C_T(1) \varepsilon_i\right) + O\left(N^{-\frac{1}{2}}\right) \end{aligned}$$

such that

$$\frac{1}{N} \sum_{i=1}^{N} \text{Var}\left(Q_{1i} + \frac{3}{T+1} R_{1i}\right) \to g_1 m_4 + g_2 \sigma_{4\varepsilon} \quad \text{as } N \to \infty \tag{133}$$

as $\frac{1}{N}\sum_{i=1}^{N} E\left(\varepsilon_{it}^4\right) \to m_4$ according to Assumption 4 (iv) and the following results

$$\begin{aligned} E\left(\left(\varepsilon_i' C_T(1)' Q_T \varepsilon_i\right)^2\right) &= \frac{(T-1)(2T-1)}{6T} E\left(\varepsilon_{it}^4\right) + \frac{(T-1)\left(2T^2 - 4T + 3\right)}{6T} \sigma_{i\varepsilon}^4 \\ E\left(\left(\varepsilon_i' C_T(1)' Q_T C_T(1) \varepsilon_i\right)^2\right) &= \frac{\left(T^2-1\right)\left(T^2+1\right)}{30T} E\left(\varepsilon_{it}^4\right) + \frac{(T-2)\left(T^2-1\right)\left(T^2+1\right)}{20T} \sigma_{i\varepsilon}^4 \\ E\left(\varepsilon_i' C_T(1)' Q_T \varepsilon_i \varepsilon_i' C_T(1)' Q_T C_T(1) \varepsilon_i\right) &= -\frac{\left(T^2-1\right)}{12} E\left(\varepsilon_{it}^4\right) - \frac{\left(T^2-1\right)(T-2)}{12} \sigma_{i\varepsilon}^4 \end{aligned}$$

These results can be found in Harris & Tzavalis (1999) p. 222. It has been checked that they are correct. We can now apply the Liapounov Central Limit Theorem to the sequence $\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(Q_{1i} + \frac{3}{T+1} R_{1i}\right)$ as both Q_{1i} and R_{1i} have bounded moments of order slightly greater than two according to the result in Lemma 4. Altogether, this proves the results in (122) and (123).

Next, we show the result in (124) by showing that $E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Q_{2i}\right|^{2}\rightarrow 0$ as $N\rightarrow\infty$. We have that

$$\begin{aligned}
E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Q_{2i}\right|^{2} &= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}E\left(Q_{2i}Q_{2j}\right)=\frac{1}{N}\sum_{i=1}^{N}E\left(Q_{2i}^{2}\right)\\
&= \frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{4}A_{T}\left(\rho\right)'Q_{T}Q_{T}A_{T}\left(\rho\right)\tau\left(\rho\right)=O\left(N^{-\frac{1}{2}}\right)
\end{aligned}$$

where the second equality sign holds as Q_{2i} is independent across i with mean zero as ε_{i} and ε_{i0} are independent with means zero and the last equality sign holds as $\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{4}=O\left(1\right)$, $\tau\left(\rho\right)=O\left(N^{\frac{1}{2}}\right)$ and $A_{T}\left(\rho\right)'Q_{T}=O\left(N^{-\frac{1}{2}}\right)$. Thus, $E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Q_{2i}\right|^{2}\rightarrow 0$ as $N\rightarrow\infty$. In a similar manner we show the result in (126) by showing that $E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}R_{3i}\right|^{2}\rightarrow 0$ as $N\rightarrow\infty$. We have that

$$\begin{aligned}
E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}R_{3i}\right|^{2} &= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}E\left(R_{3i}R_{3j}\right)=\frac{1}{N}\sum_{i=1}^{N}E\left(R_{3i}^{2}\right)\\
&= \frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{4}A_{T}\left(\rho\right)'Q_{T}C_{T}\left(\rho\right)C_{T}\left(\rho\right)'Q_{T}A_{T}\left(\rho\right)\tau\left(\rho\right)=O\left(N^{-\frac{1}{2}}\right)
\end{aligned}$$

where we have used that $\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{4}=O\left(1\right)$, $\tau\left(\rho\right)=O\left(N^{\frac{1}{2}}\right)$, $C_{T}\left(\rho\right)=O\left(1\right)$ and $A_{T}\left(\rho\right)'Q_{T}=O\left(N^{-\frac{1}{2}}\right)$. Thus, $E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}R_{3i}\right|^{2}\rightarrow 0$ as $N\rightarrow\infty$. Finally, we show (126) by showing that $E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(R_{2i}-E\left(R_{2i}\right)\right)\right|^{2}\rightarrow 0$ as $N\rightarrow\infty$. We have that

$$\begin{aligned}
E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(R_{2i}-E\left(R_{2i}\right)\right)\right|^{2} &= \frac{1}{N}\sum_{i=1}^{N}\mathrm{Var}\left(R_{2i}\right)\leq\frac{1}{N}\sum_{i=1}^{N}E\left(R_{2i}^{2}\right)\\
&= \frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{4}\tau\left(\rho\right)^{2}\left(A_{T}\left(\rho\right)'Q_{T}A_{T}\left(\rho\right)\right)^{2}=O\left(N^{-1}\right)
\end{aligned}$$

where the first equality sign holds as $R_{2i}-E\left(R_{2i}\right)$ is independent across i with mean zero and the last equality sign holds as $\frac{1}{N}\sum_{i=1}^{N}\sigma_{i\varepsilon}^{4}=O\left(1\right)$, $\tau\left(\rho\right)=O\left(N^{\frac{1}{2}}\right)$ and $A_{T}\left(\rho\right)'Q_{T}A_{T}\left(\rho\right)=O\left(N^{-1}\right)$. Altogether, we have proved the results in (b1) and (b2). □

Proof of Proposition 6:

The proposition follows by the results already obtained in the previous and Lemma 12 given below.

Lemma 12 *Under Assumption 1, 2, 3, 4 and the local-to-unity sequence for ρ given by $\rho=1-c/\sqrt{N}$ for $c\geq 0$ the following result holds*

$$\frac{1}{N}\sum_{i=1}^{N}y_{i,-1}'Q_{T}\hat{\omega}_{i}\hat{\omega}_{i}'Q_{T}y_{i,-1}\xrightarrow{P}g_{1}m_{4}+g_{2}\sigma_{4\varepsilon}\quad \text{as } N\rightarrow\infty \tag{a}$$

Combining the result in Lemma 12 with (a) in Lemma 11 we have that

$$\begin{aligned}
\hat{V}_{WG} &= \left(\frac{1}{N}\sum_{i=1}^{N}y_{i,-1}'Q_{T}y_{i,-1}\right)^{-1}\frac{1}{N}\sum_{i=1}^{N}y_{i,-1}'Q_{T}\hat{\omega}_{i}\hat{\omega}_{i}'Q_{T}y_{i,-1}\left(\frac{1}{N}\sum_{i=1}^{N}y_{i,-1}'Q_{T}y_{i,-1}\right)^{-1}\\
&\xrightarrow{P}\frac{k_{1}m_{4}+k_{2}\sigma_{4\varepsilon}}{\sigma_{2\varepsilon}^{2}}\quad \text{as } N\rightarrow\infty
\end{aligned}\tag{134}$$

where k_1 and k_2 are defined in (31). Using the expressions for $\bar{t}_{WG}$ and t_{WG} given in (33) and (35) respectively we have that

$$\bar{t}_{WG} = \tilde{V}_{WG}^{-\frac{1}{2}}\sqrt{N}\left(\hat{\rho}_0 - 1\right) = \tilde{V}_{WG}^{-\frac{1}{2}}\sqrt{N}\left(\hat{\rho}_0 - \rho\right) - c\tilde{V}_{WG}^{-\frac{1}{2}} \tag{135}$$

$$t_{WG} = \hat{V}_{WG}^{-\frac{1}{2}}\sqrt{N}\left(\hat{\rho}_0 - 1\right) = \hat{V}_{WG}^{-\frac{1}{2}}\sqrt{N}\left(\hat{\rho}_0 - \rho\right) - c\hat{V}_{WG}^{-\frac{1}{2}} \tag{136}$$

where $\tilde{V}_{WG}$ is defined in (32). Using this, the results in Proposition 5 yields the results in Proposition 6.

Proof of Lemma 12:

Inserting the expression for $\hat{\omega}_i$ given by $\hat{\omega}_i = Q_T\varepsilon_i + \left(\rho - \hat{\rho}_{WG}\right)Q_T y_{i,-1}$ yields

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N} y_{i,-1}' Q_T \hat{\omega}_i \hat{\omega}_i' Q_T y_{i,-1} &= \frac{1}{N}\sum_{i=1}^{N}\left(y_{i,-1}' Q_T \varepsilon_i\right)^2 + \left(\rho - \hat{\rho}_{WG}\right)^2 \frac{1}{N}\sum_{i=1}^{N}\left(y_{i,-1}' Q_T y_{i,-1}\right)^2 \\ &\quad + 2\left(\rho - \hat{\rho}_{WG}\right)\frac{1}{N}\sum_{i=1}^{N} y_{i,-1}' Q_T y_{i,-1} y_{i,-1}' Q_T \varepsilon_i\end{aligned}$$

As $\hat{\rho}_{WG} - \rho = -\frac{3}{T+1} + o_P(1)$ by Proposition 5, we prove the result by showing that

$$\frac{1}{N}\sum_{i=1}^{N} Q_{1i}^2 + \frac{3}{T+1}\frac{1}{N}\sum_{i=1}^{N} R_{1i}^2 + \frac{6}{T+1}\frac{1}{N}\sum_{i=1}^{N} R_{1i}Q_{1i} \xrightarrow{P} g_1 m_4 + g_2\sigma_{4\varepsilon} \quad \text{as } N \to \infty \tag{137}$$

and that

$$\frac{1}{N}\sum_{i=1}^{N}\left(y_{i,-1}' Q_T \varepsilon_i\right)^2 - \frac{1}{N}\sum_{i=1}^{N} Q_{1i}^2 \xrightarrow{P} 0 \quad \text{as } N \to \infty \tag{138}$$

$$\frac{1}{N}\sum_{i=1}^{N}\left(y_{i,-1}' Q_T y_{i,-1}\right)^2 - \frac{1}{N}\sum_{i=1}^{N} R_{1i}^2 \xrightarrow{P} 0 \quad \text{as } N \to \infty \tag{139}$$

$$\frac{1}{N}\sum_{i=1}^{N} y_{i,-1}' Q_T y_{i,-1} y_{i,-1}' Q_T \varepsilon_i - \frac{1}{N}\sum_{i=1}^{N} Q_{1i}R_{1i} \xrightarrow{P} 0 \quad \text{as } N \to \infty \tag{140}$$

The result in (137) holds by Markov's Law of Large Numbers which can be applied according to (47) and (49) in Lemma 4 and (50) and (51) also in Lemma 4 together with the result in (133). Using the expressions for R_{ki} for $k = 1, 2, 3$ and Q_{ki} for $k = 1, 2$ we have

$$\begin{aligned}E\left(R_{1i}^2\right) &= O(1) \\ E\left(R_{2i}^2\right) &= E\left(\varepsilon_{i0}^4\right)\tau(\rho)^2\left(A_T(\rho)' Q_T A_T(\rho)\right)^2 = O\left(N^{-1}\right) \\ E\left(R_{3i}^2\right) &= \sigma_{i\varepsilon}^4 A_T(\rho)' Q_T C_T(\rho) C_T(\rho)' Q_T A_T(\rho) = O\left(N^{-\frac{1}{2}}\right) \\ E\left(Q_{1i}^2\right) &= O(1) \\ E\left(Q_{2i}^2\right) &= \sigma_{i\varepsilon}^4 A_T(\rho)' Q_T A_T(\rho)\tau(\rho) = O\left(N^{-\frac{1}{2}}\right)\end{aligned}$$

To prove (138) we show that $E\left|\frac{1}{N}\sum_{i=1}^{N}\left(y_{i,-1}'Q_T\varepsilon_i\right)^2-\frac{1}{N}\sum_{i=1}^{N}Q_{1i}^2\right|\to 0$ as $N\to\infty$. We have

$$\begin{aligned} E\left|\frac{1}{N}\sum_{i=1}^{N}\left(y_{i,-1}'Q_T\varepsilon_i\right)^2-\frac{1}{N}\sum_{i=1}^{N}Q_{1i}^2\right| &= E\left|\frac{1}{N}\sum_{i=1}^{N}\left(Q_{2i}^2+2Q_{1i}Q_{2i}\right)\right| \\ &\leq \frac{1}{N}\sum_{i=1}^{N}\left(E\left(Q_{2i}^2\right)+2E\left(Q_{1i}^2\right)^{\frac{1}{2}}E\left(Q_{2i}^2\right)^{\frac{1}{2}}\right) \\ &= O\left(N^{-\frac{1}{4}}\right) \end{aligned}$$

where the inequality follows by the triangle inequality and the Cauchy-Schwarz inequality and the last line holds by using the expression for Q_{1i} and Q_{2i} given above. To prove (139) we use that

$$\begin{aligned} & E\left|\frac{1}{N}\sum_{i=1}^{N}\left(y_{i,-1}'Q_Ty_{i,-1}\right)^2-\frac{1}{N}\sum_{i=1}^{N}R_{1i}^2\right| \\ = \; & E\left|\frac{1}{N}\sum_{i=1}^{N}R_{2i}^2+R_{3i}^2+2R_{1i}\left(R_{2i}+R_{3i}\right)+2R_{2i}R_{3i}\right| \\ \leq \; & \frac{1}{N}\sum_{i=1}^{N}\left(E\left(R_{2i}^2\right)+E\left(R_{3i}^2\right)+2E\left(R_{1i}^2\right)^{\frac{1}{2}}E\left(R_{2i}^2\right)^{\frac{1}{2}}+2E\left(R_{1i}^2\right)^{\frac{1}{2}}E\left(R_{3i}^2\right)^{\frac{1}{2}}+2E\left(R_{2i}^2\right)^{\frac{1}{2}}E\left(R_{3i}^2\right)^{\frac{1}{2}}\right) \\ = \; & O\left(N^{-\frac{1}{4}}\right) \end{aligned}$$

where the inequality again follows by using the triangle inequality and the Cauchy-Schwarz inequality. Thus $E\left|\frac{1}{N}\sum_{i=1}^{N}\left(y_{i,-1}'Q_Ty_{i,-1}\right)^2-\frac{1}{N}\sum_{i=1}^{N}R_{1i}^2\right|\to 0$ as $N\to\infty$. Finally, by using similar arguments we have that

$$\begin{aligned} & E\left|\frac{1}{N}\sum_{i=1}^{N}y_{i,-1}'Q_Ty_{i,-1}y_{i,-1}'Q_T\varepsilon_i-\frac{1}{N}\sum_{i=1}^{N}Q_{1i}R_{1i}\right| \\ = \; & E\left|\frac{1}{N}\sum_{i=1}^{N}Q_{1i}R_{2i}+Q_{1i}R_{3i}+Q_{2i}R_{1i}+Q_{2i}R_{2i}+Q_{2i}R_{3i}\right| \\ \leq \; & \frac{1}{N}\sum_{i=1}^{N}E\left(Q_{1i}^2\right)^{\frac{1}{2}}\left(E\left(R_{2i}^2\right)^{\frac{1}{2}}+E\left(R_{3i}^2\right)^{\frac{1}{2}}\right)+E\left(Q_{2i}^2\right)^{\frac{1}{2}}\left(E\left(R_{1i}^2\right)^{\frac{1}{2}}+E\left(R_{2i}^2\right)^{\frac{1}{2}}+E\left(R_{3i}^2\right)^{\frac{1}{2}}\right) \\ = \; & O\left(N^{-\frac{1}{4}}\right) \end{aligned}$$

Thus, $E\left|\frac{1}{N}\sum_{i=1}^{N}y_{i,-1}'Q_Ty_{i,-1}y_{i,-1}'Q_T\varepsilon_i-\frac{1}{N}\sum_{i=1}^{N}Q_{1i}R_{1i}\right|\to 0$ as $N\to\infty$. Altogether, we have obtained the desired limits and the result is proved. □

B Appendix

This appendix contains additional information on the outcome of the simulation experiments in Section 4. In the tables, the mean and standard deviation of the estimator of the autoregressive parameter are reported. In Table 7-10 the columns corresponding to Test 1 and Test 2 show the empirical rejection probabilities of the tests corresponding to the t-statistic and the normalized coefficient statistic, respectively. In the figures, graphs of the local power is shown together with plots of the empirical power from the simulation experiments. Using the same notation as in the tables, Test 1 and Test 2 in Figure 3 and 4 correspond to the t-statistic and the normalized coefficient statistic, respectively.

Figure 2: Power of the OLS test

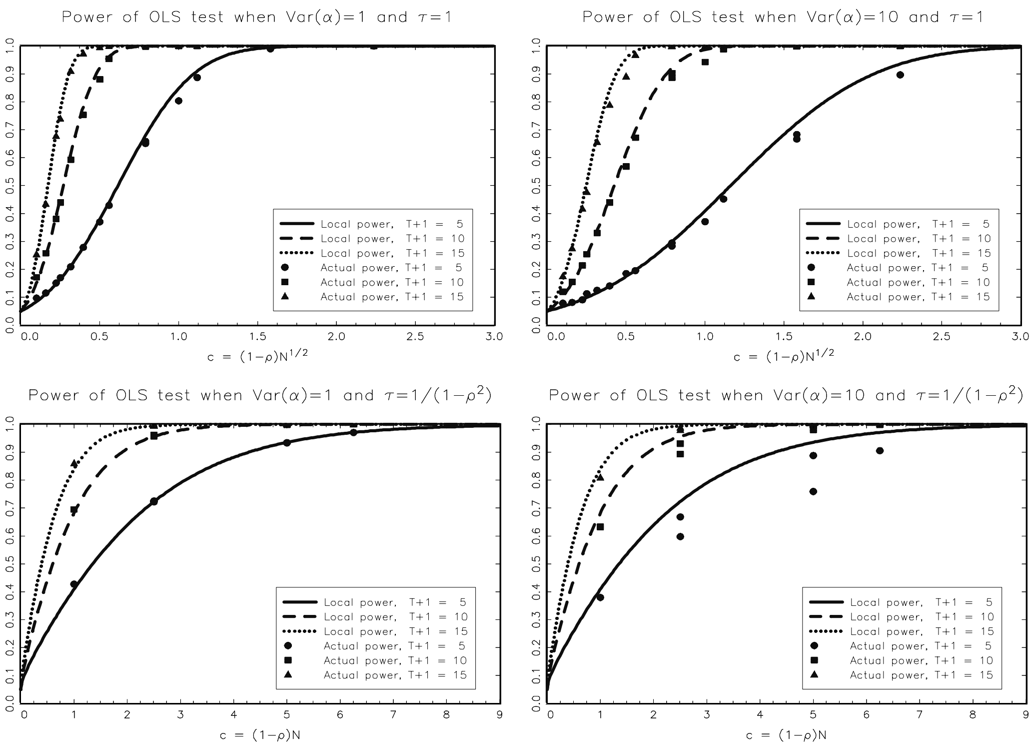


Figure 3: Power of the Breitung-Meyer tests

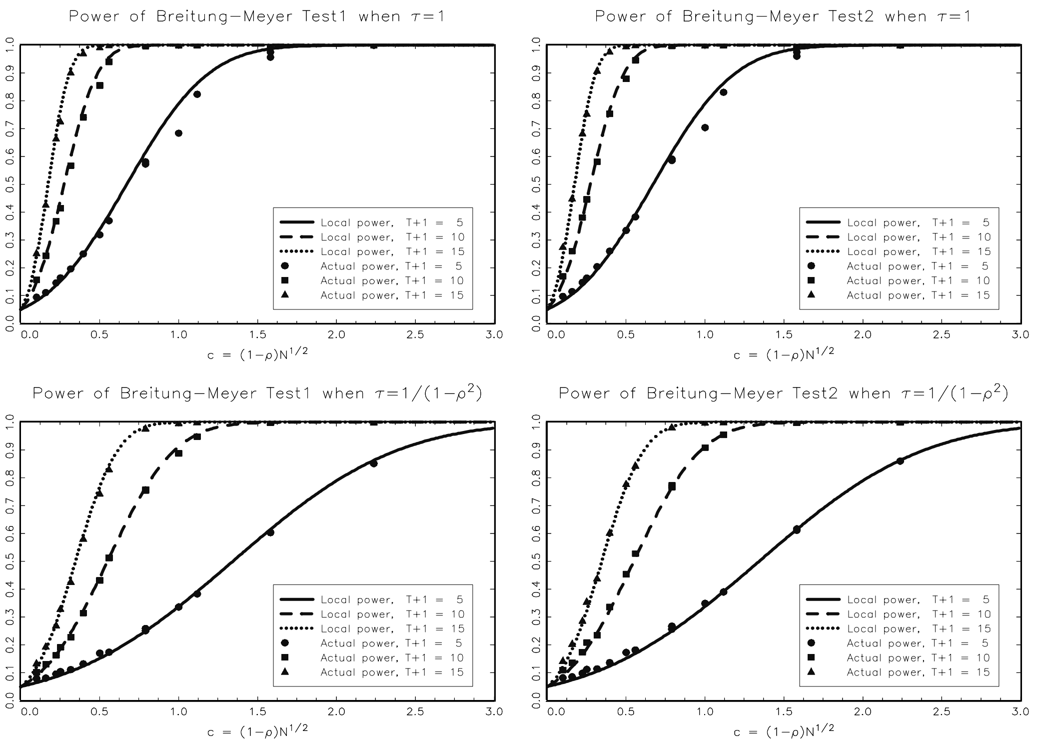


Figure 4: Power of the Harris-Tzavalis tests

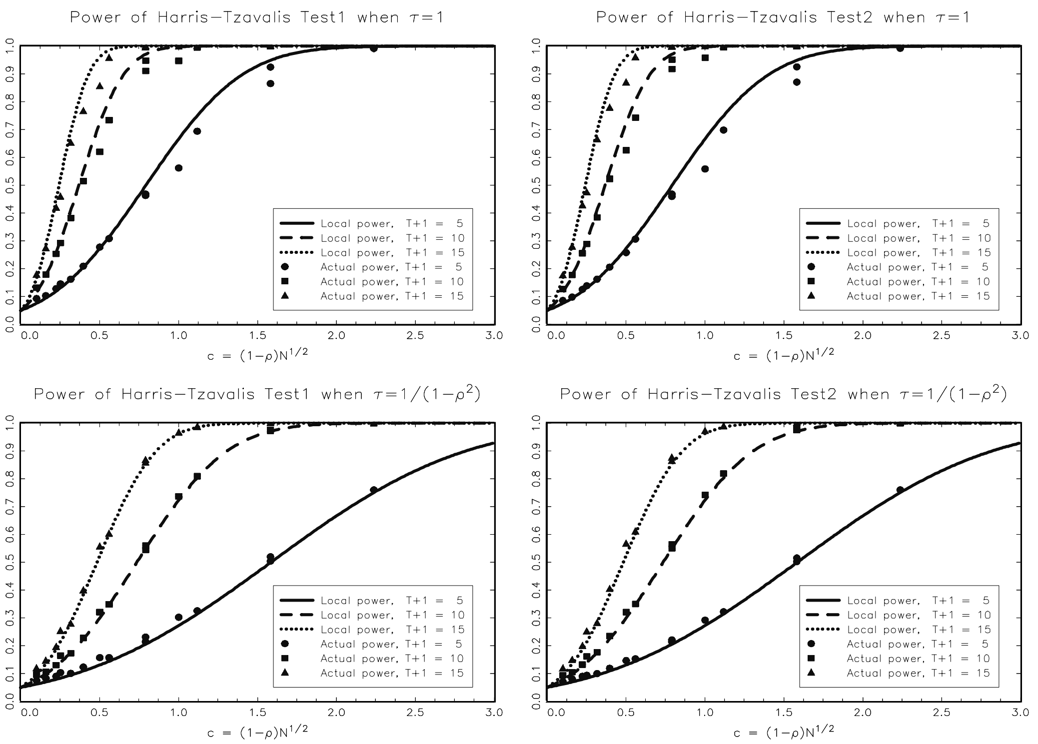


Table 3: OLS statistics when $\sigma_\alpha^2 = 1$ and $\tau(\rho) = 1$

ρ	$T+1$	N	Mean	Std. Dev.	Empirical power	Local power
0.900	5	100	0.9309	0.0283	0.8032	0.8480
0.900	5	250	0.9321	0.0178	0.9892	0.9951
0.900	5	500	0.9324	0.0128	1.0000	1.0000
0.900	5	1000	0.9324	0.0091	1.0000	1.0000
0.900	10	100	0.9226	0.0166	1.0000	1.0000
0.900	10	250	0.9233	0.0103	1.0000	1.0000
0.900	10	500	0.9238	0.0072	1.0000	1.0000
0.900	10	1000	0.9241	0.0051	1.0000	1.0000
0.900	15	100	0.9201	0.0124	1.0000	1.0000
0.900	15	250	0.9207	0.0079	1.0000	1.0000
0.900	15	500	0.9209	0.0055	1.0000	1.0000
0.900	15	1000	0.9210	0.0039	1.0000	1.0000
0.950	5	100	0.9637	0.0275	0.3694	0.3788
0.950	5	250	0.9648	0.0173	0.6504	0.6801
0.950	5	500	0.9651	0.0125	0.8868	0.9104
0.950	5	1000	0.9652	0.0089	0.9916	0.9951
0.950	10	100	0.9589	0.0153	0.8796	0.9218
0.950	10	250	0.9595	0.0095	0.9978	0.9993
0.950	10	500	0.9599	0.0066	1.0000	1.0000
0.950	10	1000	0.9602	0.0046	1.0000	1.0000
0.950	15	100	0.9573	0.0108	0.9958	0.9992
0.950	15	250	0.9579	0.0069	1.0000	1.0000
0.950	15	500	0.9581	0.0048	1.0000	1.0000
0.950	15	1000	0.9582	0.0034	1.0000	1.0000
0.990	5	100	0.9913	0.0270	0.0968	0.0842
0.990	5	250	0.9924	0.0169	0.1146	0.1108
0.990	5	500	0.9927	0.0122	0.1502	0.1475
0.990	5	1000	0.9928	0.0087	0.2088	0.2119
0.990	10	100	0.9905	0.0142	0.1712	0.1509
0.990	10	250	0.9911	0.0088	0.2574	0.2493
0.990	10	500	0.9915	0.0062	0.3798	0.3914
0.990	10	1000	0.9917	0.0043	0.5926	0.6147
0.990	15	100	0.9905	0.0096	0.2528	0.2475
0.990	15	250	0.9910	0.0061	0.4340	0.4511
0.990	15	500	0.9912	0.0042	0.6770	0.6941
0.990	15	1000	0.9912	0.0030	0.9104	0.9191
1.000	5	100	0.9979	0.0318	0.0568	0.0500
1.000	5	250	0.9991	0.0200	0.0538	0.0500
1.000	5	500	0.9996	0.0141	0.0546	0.0500
1.000	5	1000	0.9998	0.0099	0.0520	0.0500
1.000	10	100	0.9989	0.0152	0.0638	0.0500
1.000	10	250	0.9994	0.0096	0.0564	0.0500
1.000	10	500	0.9998	0.0066	0.0548	0.0500
1.000	10	1000	1.0000	0.0046	0.0454	0.0500
1.000	15	100	0.9992	0.0098	0.0554	0.0500
1.000	15	250	0.9997	0.0062	0.0578	0.0500
1.000	15	500	0.9999	0.0043	0.0502	0.0500
1.000	15	1000	1.0000	0.0031	0.0442	0.0500

Table 4: OLS statistics when $\sigma_{\alpha}^{2}=10$ and $\tau(\rho)=1$

ρ	$T+1$	N	Mean	Std. Dev.	Empirical power	Local power
0.900	5	100	0.9822	0.0133	0.3702	0.4088
0.900	5	250	0.9827	0.0084	0.6660	0.7228
0.900	5	500	0.9828	0.0060	0.8962	0.9354
0.900	5	1000	0.9828	0.0043	0.9930	0.9977
0.900	10	100	0.9753	0.0081	0.9430	0.9871
0.900	10	250	0.9757	0.0050	0.9994	1.0000
0.900	10	500	0.9759	0.0036	1.0000	1.0000
0.900	10	1000	0.9760	0.0025	1.0000	1.0000
0.900	15	100	0.9721	0.0064	0.9998	1.0000
0.900	15	250	0.9725	0.0040	1.0000	1.0000
0.900	15	500	0.9727	0.0028	1.0000	1.0000
0.900	15	1000	0.9727	0.0020	1.0000	1.0000
0.950	5	100	0.9902	0.0137	0.1840	0.1742
0.950	5	250	0.9907	0.0087	0.2822	0.2992
0.950	5	500	0.9907	0.0062	0.4516	0.4746
0.950	5	1000	0.9907	0.0044	0.6828	0.7228
0.950	10	100	0.9852	0.0083	0.5674	0.6147
0.950	10	250	0.9856	0.0051	0.8864	0.9218
0.950	10	500	0.9858	0.0037	0.9888	0.9964
0.950	10	1000	0.9859	0.0026	1.0000	1.0000
0.950	15	100	0.9826	0.0064	0.8906	0.9563
0.950	15	250	0.9830	0.0040	0.9990	0.9999
0.950	15	500	0.9831	0.0028	1.0000	1.0000
0.950	15	1000	0.9831	0.0020	1.0000	1.0000
0.990	5	100	0.9975	0.0141	0.0780	0.0664
0.990	5	250	0.9980	0.0089	0.0810	0.0776
0.990	5	500	0.9980	0.0064	0.0908	0.0920
0.990	5	1000	0.9980	0.0046	0.1244	0.1155
0.990	10	100	0.9960	0.0086	0.1190	0.1043
0.990	10	250	0.9964	0.0053	0.1552	0.1509
0.990	10	500	0.9966	0.0038	0.2128	0.2180
0.990	10	1000	0.9967	0.0027	0.3292	0.3372
0.990	15	100	0.9954	0.0064	0.1750	0.1650
0.990	15	250	0.9958	0.0041	0.2746	0.2795
0.990	15	500	0.9959	0.0028	0.4168	0.4424
0.990	15	1000	0.9959	0.0020	0.6554	0.6831
1.000	5	100	0.9979	0.0318	0.0568	0.0500
1.000	5	250	0.9991	0.0200	0.0538	0.0500
1.000	5	500	0.9996	0.0141	0.0546	0.0500
1.000	5	1000	0.9998	0.0099	0.0520	0.0500
1.000	10	100	0.9989	0.0152	0.0638	0.0500
1.000	10	250	0.9994	0.0096	0.0564	0.0500
1.000	10	500	0.9998	0.0066	0.0548	0.0500
1.000	10	1000	1.0000	0.0046	0.0454	0.0500
1.000	15	100	0.9992	0.0098	0.0554	0.0500
1.000	15	250	0.9997	0.0062	0.0578	0.0500
1.000	15	500	0.9999	0.0043	0.0502	0.0500
1.000	15	1000	1.0000	0.0031	0.0442	0.0500

Table 5: OLS statistics when $\sigma_\alpha^2 = 1$ and $\tau(\rho) = 1/\left(1-\rho^2\right)$

ρ	$T+1$	N	Mean	Std. Dev.	Empirical power	Local power
0.900	5	100	0.9151	0.0200	0.9966	0.9977
0.900	5	250	0.9157	0.0126	1.0000	1.0000
0.900	5	500	0.9158	0.0091	1.0000	1.0000
0.900	5	1000	0.9159	0.0064	1.0000	1.0000
0.900	10	100	0.9150	0.0136	1.0000	1.0000
0.900	10	250	0.9154	0.0084	1.0000	1.0000
0.900	10	500	0.9158	0.0059	1.0000	1.0000
0.900	10	1000	0.9159	0.0041	1.0000	1.0000
0.900	15	100	0.9152	0.0107	1.0000	1.0000
0.900	15	250	0.9157	0.0069	1.0000	1.0000
0.900	15	500	0.9159	0.0048	1.0000	1.0000
0.900	15	1000	0.9159	0.0034	1.0000	1.0000
0.950	5	100	0.9539	0.0149	0.9338	0.9354
0.950	5	250	0.9543	0.0093	0.9998	0.9996
0.950	5	500	0.9544	0.0067	1.0000	1.0000
0.950	5	1000	0.9544	0.0047	1.0000	1.0000
0.950	10	100	0.9538	0.0101	0.9996	0.9990
0.950	10	250	0.9541	0.0063	1.0000	1.0000
0.950	10	500	0.9543	0.0044	1.0000	1.0000
0.950	10	1000	0.9544	0.0031	1.0000	1.0000
0.950	15	100	0.9539	0.0079	1.0000	1.0000
0.950	15	250	0.9543	0.0050	1.0000	1.0000
0.950	15	500	0.9544	0.0035	1.0000	1.0000
0.950	15	1000	0.9544	0.0025	1.0000	1.0000
0.990	5	100	0.9900	0.0070	0.4268	0.4088
0.990	5	250	0.9902	0.0044	0.7216	0.7228
0.990	5	500	0.9902	0.0031	0.9328	0.9354
0.990	5	1000	0.9902	0.0022	0.9982	0.9977
0.990	10	100	0.9901	0.0047	0.6946	0.6831
0.990	10	250	0.9901	0.0030	0.9566	0.9563
0.990	10	500	0.9902	0.0021	0.9984	0.9990
0.990	10	1000	0.9902	0.0015	1.0000	1.0000
0.990	15	100	0.9901	0.0037	0.8598	0.8416
0.990	15	250	0.9901	0.0024	0.9944	0.9944
0.990	15	500	0.9902	0.0017	1.0000	1.0000
0.990	15	1000	0.9902	0.0012	1.0000	1.0000

Table 6: OLS statistics when $\sigma_{\alpha}^{2} = 10$ and $\tau(\rho) = 1/\left(1-\rho^{2}\right)$

ρ	$T+1$	N	Mean	Std. Dev.	Empirical power	Local power
0.900	5	100	0.9650	0.0126	0.8826	0.9977
0.900	5	250	0.9654	0.0079	0.9972	1.0000
0.900	5	500	0.9655	0.0057	1.0000	1.0000
0.900	5	1000	0.9655	0.0041	1.0000	1.0000
0.900	10	100	0.9648	0.0082	0.9978	1.0000
0.900	10	250	0.9652	0.0051	1.0000	1.0000
0.900	10	500	0.9654	0.0036	1.0000	1.0000
0.900	10	1000	0.9654	0.0025	1.0000	1.0000
0.900	15	100	0.9650	0.0066	1.0000	1.0000
0.900	15	250	0.9653	0.0042	1.0000	1.0000
0.900	15	500	0.9654	0.0029	1.0000	1.0000
0.900	15	1000	0.9655	0.0021	1.0000	1.0000
0.950	5	100	0.9744	0.0110	0.7588	0.9354
0.950	5	250	0.9746	0.0069	0.9806	0.9996
0.950	5	500	0.9747	0.0050	1.0000	1.0000
0.950	5	1000	0.9746	0.0036	1.0000	1.0000
0.950	10	100	0.9742	0.0073	0.9788	0.9990
0.950	10	250	0.9744	0.0045	1.0000	1.0000
0.950	10	500	0.9746	0.0032	1.0000	1.0000
0.950	10	1000	0.9746	0.0023	1.0000	1.0000
0.950	15	100	0.9743	0.0058	0.9994	1.0000
0.950	15	250	0.9746	0.0037	1.0000	1.0000
0.950	15	500	0.9746	0.0026	1.0000	1.0000
0.950	15	1000	0.9746	0.0018	1.0000	1.0000
0.990	5	100	0.9915	0.0065	0.3790	0.4088
0.990	5	250	0.9917	0.0040	0.6682	0.7228
0.990	5	500	0.9917	0.0029	0.8878	0.9354
0.990	5	1000	0.9916	0.0020	0.9922	0.9977
0.990	10	100	0.9915	0.0043	0.6324	0.6831
0.990	10	250	0.9916	0.0027	0.9308	0.9563
0.990	10	500	0.9916	0.0019	0.9960	0.9990
0.990	10	1000	0.9917	0.0013	1.0000	1.0000
0.990	15	100	0.9915	0.0034	0.8086	0.8416
0.990	15	250	0.9916	0.0022	0.9878	0.9944
0.990	15	500	0.9916	0.0015	1.0000	1.0000
0.990	15	1000	0.9916	0.0011	1.0000	1.0000

Table 7: Breitung-Meyer statistics when $\tau(\rho) = 1$

ρ	$T+1$	N	Mean	Std. Dev.	Test 1	Test 2	Local power
0.900	5	100	0.9081	0.0440	0.6838	0.7038	0.7895
0.900	5	250	0.9093	0.0278	0.9566	0.9598	0.9871
0.900	5	500	0.9099	0.0195	0.9992	0.9996	0.9999
0.900	5	1000	0.9102	0.0139	1.0000	1.0000	1.0000
0.900	10	100	0.9100	0.0201	0.9994	0.9998	1.0000
0.900	10	250	0.9107	0.0126	1.0000	1.0000	1.0000
0.900	10	500	0.9113	0.0089	1.0000	1.0000	1.0000
0.900	10	1000	0.9115	0.0062	1.0000	1.0000	1.0000
0.900	15	100	0.9112	0.0141	1.0000	1.0000	1.0000
0.900	15	250	0.9119	0.0089	1.0000	1.0000	1.0000
0.900	15	500	0.9122	0.0062	1.0000	1.0000	1.0000
0.900	15	1000	0.9123	0.0044	1.0000	1.0000	1.0000
0.950	5	100	0.9503	0.0428	0.3180	0.3336	0.3372
0.950	5	250	0.9515	0.0270	0.5720	0.5848	0.6147
0.950	5	500	0.9520	0.0190	0.8232	0.8300	0.8630
0.950	5	1000	0.9524	0.0135	0.9748	0.9750	0.9871
0.950	10	100	0.9513	0.0185	0.8548	0.8794	0.9123
0.950	10	250	0.9520	0.0117	0.9966	0.9974	0.9990
0.950	10	500	0.9525	0.0082	1.0000	1.0000	1.0000
0.950	10	1000	0.9528	0.0057	1.0000	1.0000	1.0000
0.950	15	100	0.9518	0.0124	0.9926	0.9960	0.9991
0.950	15	250	0.9525	0.0078	1.0000	1.0000	1.0000
0.950	15	500	0.9528	0.0055	1.0000	1.0000	1.0000
0.950	15	1000	0.9529	0.0039	1.0000	1.0000	1.0000
0.990	5	100	0.9879	0.0417	0.0936	0.0966	0.0808
0.990	5	250	0.9890	0.0263	0.1108	0.1136	0.1043
0.990	5	500	0.9896	0.0185	0.1452	0.1468	0.1363
0.990	5	1000	0.9899	0.0131	0.1954	0.2036	0.1921
0.990	10	100	0.9888	0.0171	0.1566	0.1686	0.1480
0.990	10	250	0.9894	0.0108	0.2424	0.2590	0.2432
0.990	10	500	0.9899	0.0075	0.3670	0.3798	0.3809
0.990	10	1000	0.9902	0.0053	0.5672	0.5812	0.5997
0.990	15	100	0.9892	0.0109	0.2526	0.2744	0.2448
0.990	15	250	0.9898	0.0069	0.4258	0.4490	0.4457
0.990	15	500	0.9900	0.0048	0.6646	0.6826	0.6873
0.990	15	1000	0.9901	0.0034	0.9020	0.9074	0.9149
1.000	5	100	0.9978	0.0414	0.0622	0.0634	0.0500
1.000	5	250	0.9989	0.0261	0.0542	0.0570	0.0500
1.000	5	500	0.9995	0.0184	0.0550	0.0574	0.0500
1.000	5	1000	0.9998	0.0131	0.0504	0.0526	0.0500
1.000	10	100	0.9987	0.0167	0.0624	0.0696	0.0500
1.000	10	250	0.9993	0.0106	0.0600	0.0672	0.0500
1.000	10	500	0.9998	0.0074	0.0486	0.0524	0.0500
1.000	10	1000	1.0001	0.0051	0.0456	0.0470	0.0500
1.000	15	100	0.9991	0.0105	0.0582	0.0626	0.0500
1.000	15	250	0.9997	0.0066	0.0530	0.0620	0.0500
1.000	15	500	0.9999	0.0046	0.0476	0.0536	0.0500
1.000	15	1000	1.0000	0.0033	0.0458	0.0468	0.0500

Table 8: Breitung-Meyer statistics when $\tau(\rho) = 1/\left(1-\rho^2\right)$

ρ	$T+1$	N	Mean	Std. Dev.	Test 1	Test 2	Local power
0.900	5	100	0.9478	0.0430	0.3358	0.3490	0.3372
0.900	5	250	0.9489	0.0271	0.6036	0.6168	0.6147
0.900	5	500	0.9495	0.0191	0.8512	0.8596	0.8630
0.900	5	1000	0.9499	0.0136	0.9814	0.9836	0.9871
0.900	10	100	0.9485	0.0188	0.8878	0.9084	0.9123
0.900	10	250	0.9493	0.0116	0.9982	0.9980	0.9990
0.900	10	500	0.9497	0.0083	1.0000	1.0000	1.0000
0.900	10	1000	0.9499	0.0058	1.0000	1.0000	1.0000
0.900	15	100	0.9488	0.0127	0.9966	0.9982	0.9991
0.900	15	250	0.9496	0.0081	1.0000	1.0000	1.0000
0.900	15	500	0.9499	0.0056	1.0000	1.0000	1.0000
0.900	15	1000	0.9500	0.0039	1.0000	1.0000	1.0000
0.950	5	100	0.9728	0.0423	0.1696	0.1720	0.1509
0.950	5	250	0.9739	0.0267	0.2580	0.2672	0.2493
0.950	5	500	0.9745	0.0188	0.3830	0.3896	0.3914
0.950	5	1000	0.9749	0.0134	0.6032	0.6110	0.6147
0.950	10	100	0.9736	0.0177	0.4314	0.4540	0.4424
0.950	10	250	0.9743	0.0110	0.7586	0.7722	0.7663
0.950	10	500	0.9748	0.0078	0.9474	0.9544	0.9563
0.950	10	1000	0.9750	0.0054	0.9984	0.9986	0.9990
0.950	15	100	0.9739	0.0115	0.7438	0.7772	0.7703
0.950	15	250	0.9746	0.0073	0.9762	0.9806	0.9832
0.950	15	500	0.9749	0.0051	0.9998	0.9998	0.9999
0.950	15	1000	0.9750	0.0036	1.0000	1.0000	1.0000
0.990	5	100	0.9928	0.0417	0.0780	0.0810	0.0640
0.990	5	250	0.9939	0.0263	0.0800	0.0846	0.0734
0.990	5	500	0.9945	0.0185	0.0944	0.0972	0.0852
0.990	5	1000	0.9948	0.0131	0.1108	0.1134	0.1043
0.990	10	100	0.9937	0.0169	0.1006	0.1090	0.0893
0.990	10	250	0.9943	0.0106	0.1298	0.1360	0.1209
0.990	10	500	0.9948	0.0074	0.1618	0.1728	0.1650
0.990	10	1000	0.9950	0.0052	0.2266	0.2336	0.2432
0.990	15	100	0.9941	0.0107	0.1328	0.1414	0.1214
0.990	15	250	0.9947	0.0067	0.1928	0.2030	0.1865
0.990	15	500	0.9949	0.0047	0.2694	0.2860	0.2815
0.990	15	1000	0.9950	0.0033	0.4260	0.4382	0.4457

Table 9: Harris-Tzavalis statistics when $\tau(\rho) = 1$

ρ	$T+1$	N	Mean	Std. Dev.	Test 1	Test 2	Local power
0.900	5	100	0.2960	0.0585	0.5624	0.5580	0.6665
0.900	5	250	0.2970	0.0373	0.8654	0.8704	0.9491
0.900	5	500	0.2971	0.0261	0.9906	0.9916	0.9986
0.900	5	1000	0.2974	0.0183	1.0000	1.0000	1.0000
0.900	10	100	0.5958	0.0326	0.9474	0.9584	0.9977
0.900	10	250	0.5967	0.0205	0.9998	0.9998	1.0000
0.900	10	500	0.5974	0.0144	1.0000	1.0000	1.0000
0.900	10	1000	0.5975	0.0100	1.0000	1.0000	1.0000
0.900	15	100	0.7010	0.0229	0.9978	0.9988	1.0000
0.900	15	250	0.7020	0.0143	1.0000	1.0000	1.0000
0.900	15	500	0.7023	0.0101	1.0000	1.0000	1.0000
0.900	15	1000	0.7024	0.0071	1.0000	1.0000	1.0000
0.950	5	100	0.3427	0.0585	0.2768	0.2574	0.2718
0.950	5	250	0.3438	0.0375	0.4630	0.4590	0.4983
0.950	5	500	0.3439	0.0261	0.6938	0.6976	0.7502
0.950	5	1000	0.3443	0.0183	0.9240	0.9254	0.9491
0.950	10	100	0.6399	0.0318	0.6200	0.6248	0.7231
0.950	10	250	0.6407	0.0201	0.9102	0.9168	0.9708
0.950	10	500	0.6415	0.0141	0.9950	0.9964	0.9996
0.950	10	1000	0.6416	0.0098	1.0000	1.0000	1.0000
0.950	15	100	0.7416	0.0220	0.8564	0.8686	0.9618
0.950	15	250	0.7426	0.0138	0.9964	0.9972	0.9999
0.950	15	500	0.7429	0.0098	1.0000	1.0000	1.0000
0.950	15	1000	0.7430	0.0069	1.0000	1.0000	1.0000
0.990	5	100	0.3862	0.0584	0.0924	0.0852	0.0753
0.990	5	250	0.3873	0.0375	0.1034	0.0968	0.0940
0.990	5	500	0.3875	0.0261	0.1276	0.1248	0.1188
0.990	5	1000	0.3880	0.0183	0.1622	0.1618	0.1614
0.990	10	100	0.6852	0.0309	0.1306	0.1256	0.1156
0.990	10	250	0.6860	0.0195	0.1784	0.1774	0.1743
0.990	10	500	0.6867	0.0137	0.2528	0.2548	0.2596
0.990	10	1000	0.6868	0.0095	0.3822	0.3836	0.4090
0.990	15	100	0.7853	0.0209	0.1766	0.1756	0.1682
0.990	15	250	0.7863	0.0132	0.2728	0.2784	0.2863
0.990	15	500	0.7866	0.0093	0.4182	0.4270	0.4535
0.990	15	1000	0.7866	0.0066	0.6524	0.6636	0.6971
1.000	5	100	0.3980	0.0584	0.0634	0.0560	0.0500
1.000	5	250	0.3991	0.0375	0.0594	0.0598	0.0500
1.000	5	500	0.3993	0.0261	0.0574	0.0564	0.0500
1.000	5	1000	0.3998	0.0183	0.0552	0.0548	0.0500
1.000	10	100	0.6983	0.0306	0.0640	0.0588	0.0500
1.000	10	250	0.6991	0.0193	0.0608	0.0586	0.0500
1.000	10	500	0.6998	0.0135	0.0558	0.0548	0.0500
1.000	10	1000	0.6999	0.0094	0.0500	0.0502	0.0500
1.000	15	100	0.7987	0.0206	0.0626	0.0618	0.0500
1.000	15	250	0.7997	0.0130	0.0562	0.0550	0.0500
1.000	15	500	0.8000	0.0092	0.0532	0.0534	0.0500
1.000	15	1000	0.8000	0.0065	0.0490	0.0510	0.0500

Table 10: Harris-Tzavalis statistics when $\tau(\rho) = 1/\left(1-\rho^2\right)$

ρ	$T+1$	N	Mean	Std. Dev.	Test 1	Test 2	Local power
0.900	5	100	0.9375	0.0583	0.3014	0.2916	0.2718
0.900	5	250	0.9382	0.0368	0.5192	0.5148	0.4983
0.900	5	500	0.9388	0.0262	0.7600	0.7600	0.7502
0.900	5	1000	0.9394	0.0184	0.9460	0.9472	0.9491
0.900	10	100	0.9290	0.0316	0.7368	0.7418	0.7231
0.900	10	250	0.9299	0.0196	0.9756	0.9798	0.9708
0.900	10	500	0.9303	0.0138	0.9990	0.9992	0.9996
0.900	10	1000	0.9304	0.0098	1.0000	1.0000	1.0000
0.900	15	100	0.9246	0.0221	0.9652	0.9712	0.9618
0.900	15	250	0.9258	0.0139	1.0000	1.0000	0.9999
0.900	15	500	0.9263	0.0098	1.0000	1.0000	1.0000
0.900	15	1000	0.9264	0.0068	1.0000	1.0000	1.0000
0.950	5	100	0.9677	0.0582	0.1566	0.1464	0.1301
0.950	5	250	0.9684	0.0368	0.2306	0.2202	0.2048
0.950	5	500	0.9691	0.0262	0.3252	0.3216	0.3139
0.950	5	1000	0.9697	0.0184	0.5038	0.5020	0.4983
0.950	10	100	0.9641	0.0310	0.3202	0.3204	0.2993
0.950	10	250	0.9651	0.0192	0.5592	0.5644	0.5492
0.950	10	500	0.9655	0.0136	0.8086	0.8186	0.8040
0.950	10	1000	0.9657	0.0096	0.9722	0.9750	0.9708
0.950	15	100	0.9622	0.0214	0.5560	0.5660	0.5254
0.950	15	250	0.9634	0.0135	0.8670	0.8758	0.8546
0.950	15	500	0.9639	0.0095	0.9862	0.9882	0.9852
0.950	15	1000	0.9641	0.0066	1.0000	1.0000	0.9999
0.990	5	100	0.9916	0.0582	0.0742	0.0702	0.0616
0.990	5	250	0.9924	0.0368	0.0822	0.0802	0.0693
0.990	5	500	0.9931	0.0261	0.0898	0.0894	0.0789
0.990	5	1000	0.9938	0.0184	0.1000	0.0990	0.0940
0.990	10	100	0.9915	0.0304	0.0956	0.0934	0.0776
0.990	10	250	0.9926	0.0189	0.1048	0.1042	0.0983
0.990	10	500	0.9930	0.0133	0.1298	0.1310	0.1262
0.990	10	1000	0.9932	0.0094	0.1722	0.1754	0.1743
0.990	15	100	0.9911	0.0207	0.1182	0.1174	0.0963
0.990	15	250	0.9924	0.0130	0.1452	0.1482	0.1347
0.990	15	500	0.9928	0.0092	0.1950	0.1986	0.1892
0.990	15	1000	0.9930	0.0064	0.2782	0.2810	0.2863

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